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# THE THEORY and PRACTICE OF GAUGING,

Demonſtrated in a

## SHORT and EASY METHOD.

CONTAINING, among other PARTICULARS,

The Method of Computing Decimally; extracting the SQUARE ROOT, with the Conſtruction and Uſe of the SLIDING-RULE explain'd at large.

The Elementary Properties of the CONIC SECTIONS, and the Manner of deſcribing them in Plano.

General Principles of MENSURATION, with Theorems for Meaſuring all right-lin'd Planes, Parallelopipedons, Priſmatic Solids, the Conic Sections, Conoids, Spindles, their Segments, and Fruſtums, *&c.*

A General Propoſition for meaſuring the Hoofs or Ungula's of any CONE or PYRAMID; and a new Method for Meaſuring by Approximation.

With the APPLICATION of the preceeding Principles to Practice; and illuſtrated throughout by particular Examples.

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*Publiſhed with the particular Approbation of the Honourable* COMMISSIONERS OF EXCISE.

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Deſign'd for the Uſe of the Officers of that Revenue.

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By *ROBERT SHIRTCLIFFE.*

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# COMMISSIONERS
AND
# GOVERNOURS

Of and for the MANAGEMENT and RECEIPT of
His MAJESTY's Revenues of the
Excise, Malt, &c.
Within that Part of *Great-Britain* call'd
*ENGLAND;*

The following Treatise is, with the greatest Submission, humbly inscrib'd by

*Their most Obedient,*

*and most Faithful Servant,*

ROB. SHIRTCLIFFE.

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# PREFACE.

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*IN the following Treatiſe are laid down the Principles for the Menſuration of Magnitude, with ſuch other Parts of the Mathematicks only, as are neceſſarily leading thereto, and their Application in the Inveſtigating and Demonſtrating the ſeveral Theorems for meaſuring thoſe plain Surfaces and Solids, which more immediately belong to Gauging: The whole being originally deſign'd for the benefit of the Officers of the Exciſe, that they might not only be acquainted with the moſt eaſy and exact Methods for Practice, but be at the ſame time certain of the Truth of the Rules made uſe of.*

*Care has been taken to avoid meddling with the Methods practiſed by ſome others; or any way to be concerned in refuting their Errors; but here the only buſineſs is, to give the neceſſary Rules for the juſt and ready Practice of the Art of Gauging; ſhewing the Manner of coming at thoſe Rules, from Principles within themſelves the moſt perfect of all human Knowledge.*

*For that End, it is hoped, every thing here treated of, is explain'd in ſo eaſy a manner, as to*

*be within the reach even of Beginners, that will give that Attention which is unavoidably required in the Study of every Part of the Mathematicks: And it may with ſafety be affirm'd, that without a competent Skill in Algebra and Geometry, it is abſolutely impoſſible for any Perſon to determine, whether the Rules given by common Writers upon this Subject be true or falſe, and much leſs to make any (even the leaſt) Improvement in this Part of Science.*

*Tho' the Practice of Gauging may be allowed to be a ſufficient Qualification for every Officer: Yet, 'tis well known from common Experience, that, by far the greateſt Part of them, neither want Abilities to underſtand, or Ambition to have a reaſonable Proof of the Truth of the Rules and Methods on which the Practice is founded.*

*And as no Writer hitherto has thought fit to give both the* Theory *and* Practice of Gauging *in one and the ſame Book, I hope that may be look'd on as a ſufficient Apology for my Undertaking to do what I here offer, which is divided into three Parts.*

*The firſt, as an Introduction to the other Parts, containing the Method of computing decimally, by the Pen, with all the uſeful Contractions: And by the Sliding-Rule with its Conſtructions, and a very full and particular Account of its Uſe, together with the Reaſon of the ſeveral Operations made by each Method.*

*In*

*In the ſecond Part, there are ſome of the primary Principles of plain Geometry, with thoſe of the Conic Sections, as preparatory to what follows: Where ſuch general Propoſitions, for the Meaſuring of Plane Surfaces and Solids, are laid down, that the Rules for moſt of the common ones, are deduced by way of Corollaries.*

*You have here particular Rules for Menſuration of all right-lined Plane Surfaces, Circles, Conic Sections, and for Solids, ſuch as Parallelopipedons, Priſms, Conoids, Spindles, with the Segments and Fruſtums of theſe: Some of theſe perhaps were not publiſhed before, ſuch I ſuppoſe, are the Series for the Circle (in Pag. 100.) for the Hyperbola (in Pag. 109.) and the Theorems for meaſuring the Circular, Elliptic, and Hyperbolic Spindles (in Pag. 156.) Then are given the Inveſtigation of Theorems, for reducing theſe Solids nearly into Cylinders of the ſame Length: From whence is deduced, occaſionally, the common fixed Multipliers, and ſhewn how far they may be depended upon, in the Practice of Gauging. Next, a general Theorem is given, for the Menſuration of all Pyramidical or Conic Hoofs, and the ſame is accompliſh'd in the three Hoofs of an upright Cone, and that of the Square Pyramid: On this depends the method of Gauging an inclined Pyramidical Conical Tun.*

*Then the Method is ſhewn of Meaſuring by Approximation, which, if brought into Uſe,*

*would render the Art of Gauging far more compleat, than it has been hitherto: For by this Method, the Meaſure of any Caſk, Tun, Copper, or Still may be found within any deſired Degree of Truth, and that by only taking a ſufficient Number of Dimenſions, which will ever be leſs than what is required in the common methods of gauging Coppers, and the Concluſions will be much nearer the Truth. But for Caſks, there is, beſides the uſual* Data, *required only a Diameter in the Middle between the Head and the Bung; from which the Form of the Caſk, in ſome Degree, is determined.*

*This perhaps may at firſt be look'd upon as a Deſign to introduce Novelty in a Matter ſufficiently confirm'd by Experience, and by that means be rejected as a ſpeculative Nicety.*

*But if it be conſidered that the preſent Practice of Gauging depends upon aſſigning ſome known Form to the Tun or Caſk to be gauged; and without aſſuming that Form, we cannot make one Step towards computing its Meaſure: So that if we are not exact in our Aſſumption, it is not poſſible we ſhould be exact in its Content. And it may be with truth affirm'd, that there never was, ſtrictly ſpeaking, a Caſk in the Shape of any one of thoſe Solid Figures whoſe Names they bear; ſince it never could happen ſo, unleſs perfectly by Accident: The Maker having no thought of deſigning them as ſuch; and beſides, ſhould a Caſk approach nearly*

*nearly in Figure to ſuch a Solid, yet the Officer has no manner of Rule to aſſiſt him in aſcertaining the ſame; ſo that it is but mere gueſſing at beſt.*

*But by the Method I here propoſe, there is always a Certainty, either to have the Content exactly true, or exceedingly near the Truth: neither is there any Difficulty in taking the fourth Dimenſion required among the* Data; *and the Operation will be very eaſy by the Sliding-Ruler.*

*The third Part of this Book is wholly taken up with the Practice of Gauging; and is only a more full and particular Illuſtration of what was delivered in the Second Part: The Meaſure here being eſtimated by Gallons or Buſhels, which, there, were Cubical Inches. Here is ſhewn the Manner of Caſk-Gauging; Firſt, on the Suppoſition of a Caſk having a known Form: Secondly, Without any Regard to the Form, by the Help of a Fourth Dimenſion taken; ſo that every one may follow that which ſeems to him beſt.*

*Then the Practice of Ullaging Caſks ſtanding or lying is delivered in a Method, which will be found more univerſal and exact, than what is given for that Purpoſe, by the Line mark'd* Seg. ſta. *or* Seg. ly. (*which ſignifies* Segments ſtanding, *or* Segments lying) *on the Sliding-Ruler; which certainly can ſerve but one ſort of Caſks, and that muſt be ſimilar to the*

*Caſk*

*Caſk from whence the Lines themſelves were made.*

*Afterwards follows the ſeveral Rules and Precepts for Gauging Tuns, Coppers, Stills, Ciſterns, &c. with Examples at large to each. And that nothing might be wanting on my Part to render this Treatiſe as compleat as poſſibly I could, I have carefully conſulted all that has been wrote hitherto upon the Subject, and particularly* J. Kepler, P. Guildin, J. Wallis, W. Jones, —— Sharp, *and* J. Mat. Haſius; *being the moſt conſiderable Authors who have delivered any thing to the purpoſe about this Affair, and at the ſame time have demonſtrated the Rules they gave.*

*From theſe I have ſelected whatever was thought might conduce either to the Improvement of the Practice of Gauging, or that might make the Reaſon of the Methods already given more eaſy to be comprehended.*

*And as to the Way taken in finding out the ſeveral Rules herein delivered, I have not ſcrupled to make uſe of either the Arithmetic of Infinites, or the Method of Increments, according as I imagined the one or the other would render the Inveſtigation the moſt eaſy. In this I have followed the Example of one of the ingenious Authors above-mentioned, whoſe Judgment in theſe Matters will never be doubted, he having deduced the Solutions of ſeveral*

*Pro-*

*Propoſitions in his* Synopſis *after the ſame manner.*

*I hope the Reader will favourably paſs over any Inaccuracy of Style he may poſſibly meet with, and excuſe ſuch Errors as too often occur in Subjects of this Nature.*

October 30, 1740.
*Portland-ſtreet*, Corner
of *Mortimer-ſtreet*,
near *Oxford-market*.

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* The Theorem, *p.* 272. is deduced from *Cor.* 2. *Page* 191. thus, let MN*nm* (*Fig.* 74.) be any Part of the Fruſtum of a Sphere, and $m'n'$ a Diameter thereof, in the Middle betwixt the extreme ones MN, *mn*, continue *nm*, and thro' $m'$ draw a Line perpendicular to *mn*, MN, let it meet the firſt in R, the other in *q*, and the Circumference in P; alſo put $mn=y$, $MN=b$, $Oo=l$, $Rq=l$, $qP=v$, and $m'n'=m$ the Mid. Diam. then by *Corol.* 2. *Page* 191. the Meaſure of $MmnN=\overline{y^2+4m^2+b^2}\times\frac{al}{6}$. Now 'tis manifeſt from 3. *Eucl.* 36. $Rn\times Rm=RP\times m'P$; alſo by the 35 of the ſame *Elements*, $m'q\times qP=Mq\times qN$; but $Rn=y+\frac{m-y}{2}=\frac{m+y}{2}$, $Rm=\frac{m-y}{2}$, $RP=l+o$, $Rm'=\frac{l}{2}$, alſo $m'q=\frac{l}{2}$, $qp=v$, $Mq=\frac{b-m}{2}$, $qN=m+\frac{b-m}{2}=\frac{b+m}{2}$, hence the above Equations give $\frac{m+y}{2}\times\frac{m-y}{2}=\frac{m^2-y^2}{4}=l+v\times\frac{l}{2}$ and $\frac{b^2-m^2}{4}=v\times\frac{l}{2}$ the latter taken from the former leaves $\frac{2m^2-b^2-y^2}{4}=\frac{l^2}{2}$, therefore $4m^2=4l^2+2b^2+y^2$, put this for $4m^2$ in the above Expreſſion, then we have the Meaſure of $MmnN=\overline{b^2+y^2+4l^2+2b^2+2y^2}\times\frac{al}{6}=\overline{\frac{b^2+y^2}{2}+\frac{2l^2}{3}}\times al$ in cubic Inches, or $\overline{\frac{b^2+y^2}{2}\ \frac{2l^2}{3}}\times l\times,0034$. *Q. E. O.*

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THoſe that are inclined to have the Sliding-Rule, as conſtructed *Page* 240. may have it accurately made by the ingenious Mathematical Inſtrument-makers, Mr. *John Coggs* and Mr. *William Wyeth*, near St. *Dunſtan*'s Church in *Fleetſtreet*.

# ERRATA.

PAge 3. *line* 24, *for* 4, three, *read* 3. four. *P.* 14. *l.* 28. *r.* $\frac{p}{10}+\frac{s}{100}$. *p.* 14. *l.* 29. *for* $\frac{S}{5}$, *r.* $\frac{s}{5}$. *p.* 15. *l.* 5, 6, 7, 8, 9, 10. *for* $r+c$, *r.* $r+t$. *l.* 6. *r.* $\frac{10r+t}{25}$. *p.* 19. *l.* 28. *for* right, *r.* left. *p.* 34. *l.* 31. for $\frac{3}{\div 0}$, $\frac{4}{40}$, *r.* $\frac{2}{20}$, $\frac{4}{20}$. *p.* 35. *l.* 26. *for* on, *r.* to. *p.* 37. *l.* 18. *for* 16. *r.* 36. *l.* 31. *r.* firſt, 1. on. *p.* 39. *l.* 9. *r.* repreſented by a. *l.* 18. *r.* diſtance. *p.* 42. *l.* 5. *r.* $\frac{nb}{ma}\times\frac{m}{n}\times\frac{b}{a}=$. *p.* 45. *l.* 19. for C, r. *c*. *for* $-2L : c$. *r.* $2L : a+L : c$. *p.* 49. *l.* 8. *for* —, *r.* =. *p.* 50. *l.* 17. *for* proportionals, *r.* Propoſitions. *p.* 51. *l.* 29. for *b*, r. *l*. *p.* 52. *l.* 4. *for* L, *r. l*. *p.* 53. *l.* 26. *dele* each of. *p.* 55. *l.* 8. for their Diviſion, *r.* Diviſors. *p.* 57. *l.* 31. *r.* ſurface. *p.* 64. *l.* 19. *for* B*a* 'tis, *r.* B and *dc* drawn 'tis, *p.* 66. *l.* 5. for AC, *r.* AP. *p.* 67. refer to Fig. 17, 18, *l.* 34. *r.* the half Sum. *p.* 71. *l.* 26. *r.* Fig. 21. *l. ult. r.* right Lines. *p.* 83. *l.* 19. *r.* an Hyperbola. *p.* 86. *l.* 6. for *v*, r. *s*. *p.* 87. *l.* 17. for $r^{m}-2$, r. $r^{m}-5$. *p.* 88. *l.* 2. for *r*. r. *n*. *l.* 16. *for* of, *r.* a. *p.* 92. *l.* 4. *for* $1\times1\times1$, *r.* $1+1+1$. *p.* 90. *l.* 20. *dele* are. *p.* 109. *l.* 10. for $mv$, r. $mv^{m}$. *p.* 110. *l.* 18. *for* foll. *r.* above. *p.* 115. *l.* 3. *for* ſquare, *r.* ſolid. *p.* 121. *l.* 33. *for* and, *r.* add. *p.* 125. *l. antepenult. for* fruſtum, *r.* conoid. *p.* 126. *l.* 6. *for* conoid, *r.* fruſtum. *p.* 128. *l.* 13. *for* 216. *r.* 816. *p.* 136. *l.* 18. *for* 2.3. *r.* 1. 3. *p.* 141. *l.* 14. *for* $rb^2$, *r.* $3b^2$. *l.* 17. *for* $z\times\sqrt{02}-4z^2$, *r.* $y\sqrt{b^2-4z^2}$. *p.* 142. *l.* 12. for $\frac{b}{r2}$, r. $\frac{b}{12}$. *p.* 144. *l.* 5. for *p*. r. *q*. *p.* 152. *l.* 7. *for* reſpectively, *r.* on *c* reſpectively. *p.* 156. *l.* 10. *for* —, *r.* +. *p.* 159 *l.* 7. for $y=24$, r. $h=24$. *p.* 165. *l.* 2. for $6m^{d}$, r $6m$. *l.* 3. for $\frac{4}{1\div}\times\frac{0}{180}$, r. $\frac{2}{15}\times\frac{00}{42}$. *p.* 173. *l.* 19. *after* iſoſceles, *r.* and the baſe circular. *p.* 176. *l* 9. *for* D, *r.* d. *p.* 183. *l.* 6. *for* Hoof, *r.* Fruſtum Hoof. *p.* 192. *l.* 13 *for* laſt, *r.* IV. *l.* 26. *r.* Axis of the Section. *p.* 193. *l.* 7. *for* WB, *r.* W = B. *l.* 8. *for* Line, *r.* Cone. *p.* 194. *l.* 25. *for* much, *r.* little. *p.* 199. *l.* 27. *for* 58. *r.* 59.

# THEORY and PRACTICE OF GAUGING.

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## PART I.

### CHAP. I.

### *Of Decimal Fractions.*

DECIMAL Fractions are of ſuch great Importance not only in Gauging, but all kind of numeral Computations, where Fractions are concerned, the Operations being performed with the ſame eaſe as by whole Numbers only; and the Sliding-Rule, on which the Practice of Gauging ſo much depends, being decimally divided, it ſeem'd unavoidable in this Treatiſe to give a ſhort Account of *Decimal Arithmetick*. But ſince in the common Occurrencies in life, the Integer is ſeldom decimally divided, for a *Pound Sterling* is divided into 20 equal Parts,

each of which is one Shilling; a Shilling into 12 equal Parts, each of which is one Penny; and each of theſe again into four Parts, each of which is one Farthing: So a Foot is divided into 12 equal Parts, each of which is one Inch; a Year into 365, each of which is one Day, and ſo on. We muſt therefore in the firſt place ſay ſomething of dividing an Integer in general; for every Whole may be conceived as diviſible into any Number of equal Parts, any Multitude of which, leſs than the Whole, is in general (properly) call'd a *Fraction*. But it is manifeſt in order to form an Idea of the Magnitude or Value of any given Number of Equal Parts of any Whole, we muſt not only know into how many Parts it is divided, but alſo how many of them we would expreſs; thence it is, that to denote any Fraction, we are obliged to uſe two Numbers, each of which has its peculiar office, the one to ſignify into *how many* equal Parts the whole is divided, called the *Denominator*; and the other the *Number of them* we want to expreſs by the Fraction, called the *Numerator*; the latter of which is ever placed above the former.

*Ex. gr.* If the Integer was divided into four Parts, then one of them is expreſſed thus $\frac{1}{4}$, two two of them thus $\frac{2}{4}=\frac{1}{2}$, and three of them thus $\frac{3}{4}$, of which 1, 2, 3 are the Numerator, and 4 the Denominator.

But if the Integer be ſuppoſed divided decimally, that is into 10, 100, 1000, *&c.* equal Parts, then any Number of theſe, leſs than the whole, is called a *Decimal Fraction*: In this way of dividing the Integer ſince the Denominator is ever 1, with a certain Number of Cyphers, therefore here to know the Value of any Fraction, we need only know

know the Numerator, and the Number of Cyphers in the Denominator, and that may be done by prefixing Cyphers to the Numerator, till they with the Numerator, be equal in Number to the Cyphers in the Denominator; thus, $\frac{5}{10}$ is expreſs'd by ,5; $\frac{5}{100}$ by ,05; and $\frac{5}{1000}$ by ,005; ſo on the contrary, from any Decimal Fraction given, we may go back and reduce it to the Form of a Vulgar one; thus, 15 is equal to $\frac{15}{100}$: 05=$\frac{5}{100}$; ,012=$\frac{12}{1000}$, *&c.* This Thought (to whomſoever we are indebted for it) is no more than a Continuation of the common Scale, which is alſo Decimal; for in that any Figure is but $\frac{1}{10}$th, $\frac{1}{100}$th, $\frac{1}{1000}$th, what it would be if placed one, or two, or three, *&c.* places more towards the left Hand; and thus by going on till we paſs the Place of Units, we then fall in with the Fractional Scale.

Thus in the Expreſſion 6543210,123456; to expreſs the Value of any particular Figure of the integral Part, we *annex* to the Figure itſelf as many Cyphers as it is diſtant from (0,) the Place of Units, ſo to expreſs the fourth Figure from 0, which here is 4, we muſt *annex* three Cyphers, and it becomes 4000; but if we would expreſs the fourth Figure in the Decimal Part from 0, we muſt *prefix* three Cyphers, and it becomes ,0004; ſo that the common Scale continued below Units is as here under.

| *&c.* | 6 | 5 | 4 | 3 | 2 | 1 | 0, | 1 | 2 | 3 | 4 | 5 | 6 | *&c.* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *Millions.* | *Hundred Thouſands.* | *Ten Thouſands.* | *Thouſands.* | *Hundreds.* | *Tens.* | *Units.* | *Tenths.* | *Hundreds.* | *Thouſands.* | *Tens of Thouſands.* | *Hundred Thouſant.* | *Millions.* | |

From what has been ſaid it appears that Fractions commonly firſt preſent themſelves in the common Form, therefore in the following Propoſition we ſhall ſhew how to reduce them into Decimals.

## PROP. I.

*Any Vulgar Fraction being given, to reduce it to a Decimal Rule.*

Annex to the Numerator a competent Number of Cyphers, let this be a Dividend, and the Denominator the Diviſor; the Quotient of this Diviſion if there is no Remainder, is the Decimal ſought: But ſhould there be a Remainder, you may approach to any degree of Exactneſs, by annexing more Cyphers.

*Examples.*

Let it be required to reduce $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{9}{16}$ to Decimals.

*Operation.*

| | |
|---|---|
| 4) ,300 (,75=$\frac{3}{4}$ | 8) 5000 (,625=$\frac{5}{8}$ |
| 28 | 48 |
| 20 | 20 |
| 20 | 16 |
| 0 | 40 |
| | 40 |
| | 0 |

16) 900000 (,5625 = $\frac{9}{16}$
80

100
96

40
32

80
80

0

The Invention of this Rule is as follows. Let $\frac{n}{d}$ be any Vulgar Fraction which is to be reduced to a Decimal, and put $x$ for the Numerator or Decimal ſought, then 'tis plain $\frac{n}{d} = \frac{x}{1000}$, *&c.* and multiplying the Equation by 1000, *&c.* we have $x = \frac{n \times 1000}{d}$, *&c.* but to multiply any Number by 1000, *&c.* is only to annex Cyphers to the Number itſelf, whence the Rule is manifeſt.

Having now ſhewn how to turn any vulgar Fraction into a Decimal, in the next place we muſt ſhew the Method of computing by them after they are ſo reduced.

## PROP. II.

*Any Number of Decimal Fractions being given, it is required to find their Sum?*

Set them under one another, as in common Arithmetick, *viz.* tenths under tenths, hundreds under hundreds, thouſands under thouſands, *&c.* then add them together as whole Numbers, and

place the decimal Point of the Sum directly under thoſe of the Terms, ſo ſhall you have the Sum required.

| *Examp.* I. | II. | III. | IV. |
|---|---|---|---|
| ,1543 | 1,4238 | 64,521 | 743, |
| ,2754 | ,5784 | 87,043 | 82,4765 |
| ,0132 | 7,2189 | 7,218 | 1549,32187 |
| ,8054 | 6,5432 | 054 | 8,0431 |
| ,7268 | ,0741 | 65,— | ,07285 |
| 1,9751 | 15,8384 | 223,836 | 2382,91432 |

*Subſtraction of Decimals.*

## PROP. III.

*Any two decimal Fractions being given, to find their Difference; or two Quantities being given, compoſed of Integers and Decimals, from the greater to ſubſtract the Leſſer.*

*Rule.* Place the Terms under one another by the ſame Rule that was given in the above Propoſition, then ſubſtract them as if Integers, placing the decimal Point directly under thoſe above, ſo ſhall you have the Difference ſought.

| *Examp.* I. | II. | III. |
|---|---|---|
| From ,5743 | 8,7284 | 460,0954 |
| Subſtract ,2579 | 1,5896 | 29.7387 |
| Remainder ,3164 | 7,1388 | 430,3567 |

IV.

9438,0000
72,5847

9365,4153

*Mul-*

## *Multiplication of Decimals.*

### PROP. IV.

*Two Quantities compoſed of Integers and Decimals, or Decimals only, being given, to find their Product.*

Set the Units Place of the Multiplier under the laſt Figure of the Multiplicand, and the reſt in order; then multiply as in common Arithmetick, always placing the firſt Figure of every Multiplication under the multiplying Figure, ſo ſhall each Figure of theſe Products when added together, be of the ſame Name with that Figure of the Multiplicand, under which they ſtand.

| *Examp.* I. | II. | III. |
|---|---|---|
| 1,305 | ,3754 | ,135272 |
| 56,3 | : : :0,754 | : : : : :0,00425 |
| 3915 | : : 15016 | : : : : : 676360 |
| 7830 : | : 18770 : | : : : : 270544 : |
| 6525 : : | 26278 : : | : : : 541088 : : |
| 73,4715 | ,2830516 | ,00057490600 |

*Corol.* Whence the Number of decimal Places in the Product, is equal to the Sum of decimal Places in the Factors.

When the Multiplicand and Multiplier together contain more Decimals than we would chuſe to have in the Product, they may be contracted to any deſired Number, and yet the Product be as true to ſo many Figures, as if the Factors had been multiplied at large; for which obſerve this *Rule*. Under that Place of the Multiplicand which you would have ſecured, ſet the Units Place of the Multiplier, and write the reſt in an inverſe Or-

der; then let each Figure in the Multiplier begin to multiply that of the Multiplicand under which it ſtands, (but ſo as to have due regard to the Increaſe that would be brought thither from the foregoing Figures) and the reſt towards the left hand as in common Multiplication, ſetting the firſt Figure of every Product under the Units Place of the Multiplier, or under that of the Multiplicand, whoſe Place is to be kept in the Product, and the Sum of theſe Products gives that ſought.

*Examples.*

Let ,248264 be multiplied by ,725234, and ſix Figures, or five Figures, or four Figures, of the Product only required.

| *For ſix Fig.* | *For five Fig.* | *For four Fig.* |
|---|---|---|
| ,248264 | ,248264 | ,248264 |
| 432527,0 | 432527,0 | 43527,0 |
| 173785 | 17378 | 1738 |
| ::4965 | : 497 | 49 |
| ::1241 | : 124 | 12 |
| ::::50 | : 5 | 1 |
| :::::7 | 1 | ,1800 |
| :::::1 | ,18005 | |
| ,180049 | | |

But if the above Factors were multiplied at large, the Product would be as in the Margin.

| | |
|---|---|
| ,248264 | |
| 0,725234 | |
| : 993056 | |
| : 744792: | |
| : 496528:: | |
| : 1241320:: | |
| : 496528::: | |
| 1737848::::: | |
| ,180049493776 | |

The

The Reaſon of the preceeding Rule may be ſhewn after this manner; let *a*, *b*, *c*, *d*, *&c.* repreſent the *Primes*, *Seconds*, *Thirds*, *Fourths*, *&c.* reſpectively of any Multiplicand, (as 24826, *&c.* where $a=2$, $b=4$, $c=8$, $d=2$, *&c.*) and *e*, *f*, *g*, *h*, *&c.* the *Primes*, *Seconds*, *Thirds*, *&c.* of the Multiplier (ſuppoſe ,7252, *&c.* then $e=7$, $f=2$, $g=5$, *&c.* here then the Multiplicand will be repreſented thus, $\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\frac{d}{10000}$ *&c.* and the Multiplier thus, $\frac{e}{10}+\frac{f}{100}+\frac{g}{1000}+\frac{h}{10000}$ *&c.* or for brevity ſake put $n=10$, and the Terms when placed in order ſtand thus,

$\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4}$ *&c.*

$0+\frac{e}{n}+\frac{f}{n^2}+\frac{g}{n^3}+\frac{h}{n^4}$ *&c.* and the Product is as follows;

| | | | | |
|---|---|---|---|---|
| : | : | : | $+\frac{ha}{n^5}$ | $+\frac{hb}{n^6}+\frac{hc}{n^7}+\frac{hd}{n^8}$ |
| : | : | $\frac{ga}{n^4}+\frac{gh}{n^5}$ | | $+\frac{gc}{n^6}+\frac{gd}{n^7}$ |
| : | $\frac{fa}{n^3}+\frac{fb}{n^4}+\frac{fc}{n^5}$ | | | $+\frac{fd}{n^6}$ |
| $\frac{ea}{n^2}+\frac{eb}{n^3}+\frac{ec}{n^4}+\frac{ed}{n^5}$ | | | | |

But if the Multiplier is placed under the Multiplicand by our Rule, they will ſtand thus,

$\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4}$, *&c.*

*&c.* $\frac{h}{n^4}+\frac{g}{n^3}+\frac{f}{n^2}+\frac{e}{n}+0$ *Units Place.*

And from hence 'tis apparent, the Sum of the Indexes of every two Terms that ſtand under one another, is exactly equal to the Index of that Term

Term of the Multiplicand under which is (o) the Place of Units, *viz.* $n^5$ the Place in this Example to be kept; and therefore in multiplying the Factors thus placed, you need only begin with that Term of the Multiplicand which is above the multiplying Term of the Multiplier; for thereby it is evident you'll have every Term of the Product to the $n^5$*th* Place, in the ſame manner as if they are multiplied in their natural order; which will appear more evident by comparing the above Example (at large) with the following one (contracted.)

$$\frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3} + \frac{d}{n^4}, \text{ \&c.}$$

$$\text{\&c. } \frac{h}{n^4} + \frac{g}{n^3} + \frac{f}{n^2} + \frac{e}{n} + \text{o } \textit{Units Place.}$$

---

$$\frac{ea}{n^2} + \frac{eb}{n^3} + \frac{ec}{n^4} + \frac{ed}{n^5}$$
$$\frac{fa}{n^3} + \frac{fb}{n^4} + \frac{fe}{n^5}$$
$$\frac{ga}{n^4} + \frac{gb}{n^5}$$
$$+ \frac{ha}{n^5}$$

---

## *Diviſion of Decimals.*

### PROP. V.

*Two Decimal Fractions being given, or two Quantities compoſed of Integers and Decimals, to find the Quotient ariſing from dividing the one by the other.*

If there are more Figures in the Dividend than in the Diviſor, divide them after the ſame manner as you do in whole Numbers; if not, ſupply the Defect by annexing Cyphers to the former; after which, proſecute the Diviſion, and cut off from the Quotient as many Figures for Decimals, as there are Decimals in the Dividend more than

the Diviſor; but when this cannot be done, you muſt prefix a ſufficient Number of Cyphers to the Quotient to make good the Defect, and place the decimal Point on the left hand, ſo ſhall you have the Anſwer, if there is no Remainder; but when that happens, you may continue the Quotient at pleaſure, by conſtantly annexing a Cypher to each Remainder.

*Example* I. Let ,13975 be divided by 43, the Work is as follows;

```
.43) ,13975 (325
     129
    -----
      107
       86
    -----
       215
       215
    -------
         0
```

Now becauſe there are five Decimals in the Dividend, and none in the Diviſor, there muſt be five Decimals in the Quotient, which is therefore ,00325.

*Examp.* II. Let 5,29125 be divided by 4,25 then we have the following Proceſs.

```
4,25) 5,29125 (1,245
      425
     -----
      1041
       850
     -----
       1912
       1700
     -------
        2125
        2125
     -------
           0
```

The

The Quotient being 1245, and becauſe there are five Decimals in the Dividend, and but two in the Diviſor, the Quotient by the Rule will have three Decimals; and from thence it is 1,245.

*Scholium.* When the Terms in Diviſion each conſiſt of many places, and but a few Terms of the Quotient required, the Work may be ſhortned after the ſame manner as before in Multiplication: For all thoſe Terms of the Multiplier, which when multiplied by the Figure laſt put in the Quotient, would give an Index more diſtant from Unity, than the Index of the loweſt Place required in the Quotient, I ſay all ſuch may be omitted by only conſidering the Increaſe to be brought from them to the firſt of the retain'd Figure: This will be plain by an *Example*. Let it be required to divide ,743268 by ,54213 and only to have ſix Figures in the Quotient; the Operation as under.

,5 | 4 | 2 | 1 | 3) ,743268 (1,37102
54213

201138
162639

38499
37949

550
542

8

Having now given the fundamental Rules of Decimal Arithmetick, it remains only that we add to what has been ſaid, *the Method of reducing any Decimal to the known Parts of Meaſure, Weight or Coin, and that is performed by the following Rule.*

Multiply

Multiply the Decimal given by a Number which is equal to the multitude of Parts in the Denomination required, that makes an Integer of the Decimal given; then cutting off from the Product as many Figures from the right hand as is equal in Number to thoſe of the Decimal given, ſo ſhall thoſe on the left hand of the ſeparating Point be the Anſwer, and the Remainder (if any) a Decimal in that Denomination.

What are the Values of ,84375 of a *Pound Sterling*, of ,75 of a *Yard*, of ,754 of a *Year?*

,84375
20 Shillings in a Pound.

16,87500
12 Pence in a Shilling.

1750
875

10,500
4 Farthings in a Penny.

2,000

,75
3 Feet in a Yard.

2,25
,12 Inches in a Foot.

3,00

754
13 Months in a Year.

9,802
4 Weeks in a Month.

3,208
7 Days in Week.

1,456

So that the firſt is 16 *s.* 10$\frac{1}{2}$ *d.* the ſecond 2 *Feet* 3 *Inch.* the third 9 *Months* 3 *Weeks* 1,$\frac{45}{100}$ *Days.* *But to find the Value of the Decimal of a Pound Sterling, the following Rule is ſufficient,* and very ready in Practice. If

If the Figure in the Place of Seconds be above 4, add 1 to twice the Figure in the Prime's Place, ſo you have the Shillings. *Secondly*, The Exceſs of the Figure in the Seconds Place above 5, or the Figure in the Seconds Place if under 5, multiplied by 10, and added to the Thirds, is Farthings, which are eaſily reduced to Pence; but if the Farthings exceed 12, ſubſtract one from them, and 2 when they exceed 37, ſo have you the neareſt Value (in *Engliſh* Money) of the Fraction given.

Mr. *Malcolm* and *Hill* direct to ſubſtract 1, when the Farthings exceed 23, and the firſt bids ſubſtract 2 when they exceed 40, both which are deficient.

*The Invention of this Rule is as follows.* Put $p$, $s$, $t$, for the Figures ſtanding in the firſt, ſecond, and third Places of the Decimal reſpectively, that is, $p$ is the Primes, $s$ the Seconds, and $t$ the Thirds; alſo put S the Shillings, and F the Farthings ſought.

Then, $\frac{p}{10}+\frac{s}{100}+\frac{t}{1000}$ is the Fraction given, but that by Suppoſition is equal to S Shillings plus F Farthings; but S Shillings is equal to $\frac{S}{20}$*ths* of a Pound, and F Farthings is equal to $\frac{F}{960}$*ths* of a Pound; hence have we this Equation,

$\frac{P}{10}+\frac{S}{100}+\frac{t}{1000}=\frac{S}{20}+\frac{F}{960}$, which multipled by 20, gives $S+\frac{F}{48}=2p+\frac{S}{5}+\frac{t}{50}$: Now $s$ divided by 5, muſt either give in the Quotient 1 or 0, (for it can't exceed 9) therefore put $q$ for 1 or 0, (the Quotient) and $r$ the Remainder, that is, let

$\frac{s}{5}=$

$\frac{s}{5} = q + \frac{r}{5}$, then $S + \frac{F}{48} = 2p + q + \frac{r}{5} + \frac{t}{50}$, thence $S = 2p + q$ the Shillings, conſequently $\frac{F}{48} = \frac{r}{5} + \frac{t}{50}$, or $F = \frac{48or + 48t}{50} = \frac{240r + 24t}{25}$, and this may be reduced to $F = 10r + c - \frac{10r + c}{25}$: Now if $10r + c$ exceed 12, then $\frac{10r + c}{52}$ is greater than $\frac{1}{2}$, and ſo $10r + c - \frac{10r + c}{25}$ is neareſt equal to $10r + c - 1$; but if $10r + c$ exceed 37, then $\frac{10r + c}{25}$ is greater then $1\frac{1}{2}$, and conſequently $10r + c - \frac{10r + c}{25}$ is neareſt to $10r + c - 2$; from whence the Rule and Correction are evident. Q. O. E.

Rules ſimilar to this might be invented from theſe Principles, to value the Decimal of any other Integer, but I cannot ſay whether or no with equal Succeſs in Simplicity of Expreſſion. Seeing that the Multiplication is often performed with greater eaſe than Diviſion, eſpecially when the Diviſor conſiſts of many Figures, therefore before we finiſh this Chapter of Decimals, it will not be amiſs to ſhew how ſuch Diviſors may be converted into equivalent Multipliers, which is done by this Rule.

Let Unity with a competent Number of Cyphers (as Decimals) annext, be divided by the Diviſor given, and the Quotient will be the Multiplier ſought: for let Q be the Quantity divided by $d$, and $m$ ſuch a Quantity that multiplying Q this Product and the former Quotient, may be equal; hence we have this Equation, $\frac{Q}{d} = m \times Q$,

or

or $m\frac{1}{d}$, or thus, $\frac{Q}{d}=Q\times\frac{1}{d}$, hence $\frac{1}{d}$ turned into a Decimal, is the Multiplier ſought.

We ſhall illuſtrate this from the Diviſors for reducing the Content of any Body in cubick Inches to Gallons.

Then $\left\{\begin{array}{l}\frac{Q}{282}=\textit{Ale Gall.}\\ \frac{Q}{231}=\textit{Wine Gall.}\\ \frac{Q}{2150,42}\ \textit{Malt Buſh.}\end{array}\right\}$ in Q.

Whence $\left\{\begin{array}{l}m=\frac{1}{282}=,003546 \text{ the Multiplier for Ale.}\\ m=\frac{1}{231}=0043289 \text{ the Multiplier for Wine.}\\ m=\frac{1}{2150,42}=0004651 \text{ the Multiplier for Malt.}\end{array}\right.$

## CHAP. II.

### *Of the Square Root.*

*Defin.* I. THE ſquare Root of any Number or Quantity is that which multiplied by it ſelf, ſhall produce the ſaid Number or Quantity. Thus, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *&c.* are reſpectively the ſquare Root of 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, *&c.* becauſe 1x1, 2x2, 3x3, 4x4, 6x6, *&c.* produce thoſe Numbers; hence this Table of ſimple ple Roots and Squares:

| *Roots.* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | *&c.* |
|---|---|---|---|---|---|---|---|---|---|---|
| *Square.* | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | *&c.* |

*Defin.* II. The ſquare Roots of Numbers are of two Sorts, *ſimple*, when they conſiſt of one Figure only;

ly; and *compound*, when of more than one; erefore the Square Root of every Square Num- r of leſs than three Figures, is Simple, and may had from the above Table; but to obtain thoſe ich are compounded, we muſt premiſe the fol- wing *Lemmas*.

## LEMMA.

*If* n *be the Number of Integral Figures in the uare Root of any Number, then* n+1 *ſhall be the umber of Figures in the Square Root of the ſame umber when two Figures are annext to it.*

For let Q be the Number whoſe Square Root $q$ nſiſts of $n$ Figures, this when two Figures are an- xt will be 100Q, but ſince $q=\sqrt{Q}$ by Suppoſi- n, therefore $100q^2=100Q$, and therefore $10q$ he Square Root of 100Q, but there is $n$ Figures $q$, and therefore $n+1$ Figures in $10q$, or in the uare Root of Q, when two Figures are annext. *E. D.*

*Corol.* 1. Hence if any Number be divided in- Parts of two Figures, each by beginning at the ght-hand, and ſo on to the Left, the Square ot will conſiſt of as many Integral Figures as are he Number of Points more by one; that is, if e the Number of Points, then P+1 is the mber of Integral Figures in the Root: Alſo the uare Root of the firſt Diviſion to the Left-hand, Square Number, or the Root of the next leſs, otherwiſe, will be the firſt Figure of the Root.

Thus let the Number be 1743268 this divided a
above ſtands thus, 1.74.32.68; from whence i
appears there will be four Integral Figures in th
Root, and the firſt Figure will be one.

It is common to point the above Number thus
1743268, and then the Points ſtand under the la
Figure that is brought down at each Operation
and we have followed this method to prevent Con
fuſion to ſuch as have been uſed to the ſame, tho
the firſt ſeems moſt natural; but 'tis probable th
laſt was choſe to avoid Miſtakes when the Root o
a mixt Number was ſought, that conſiſted of In
tegers and Decimals, becauſe of the Reſemblanc
of the Point (.) and the Decimal Mark (,)

*Corol.* 2. By the ſame way of Reaſoning it ma
be ſhewn, that if $q$ the $r$ Root of any Number
conſiſt of $n$ Integral Figures, then the Root of th
ſame when $r$ Figures are annext to it, ſhall conſi
of $n+1$ Figures. And therefore univerſally if an
Number whoſe $r$ Root is ſought, be divided fro
the Right-hand into Parcels, each of which ha
$r$ Number of Figures, the Root ſhall conſiſt of
many Figures *plus* one, as are the Number o
ſeparating Points. Thus the by-quadrate Root o
1726854396, ſhall have two Figures in it, becauſ
when divided, it ſtands thus 17-2685-4396, an
2 is the firſt Figure of the Root.

*Corol.* 3. Therefore if any Square Numbe
(whoſe Root is required) be pointed, by placin
one Point over the Units, another over the Hu
dreds, and ſo on for every other Place to th
le

left hand, then from the above Lemma, the Root will conſiſt of as many Figures as in the Number of Points placed above the ſame.

### LEMMA II.

If $a^2$ denote any ſquare Number wanting the two laſt Figures, and $b$ be put for them, then ſhall ($e$) the remaining Figure of the Root be $\frac{b}{20a+e}$, or $e=2a)\frac{b}{10}$ (nearly: for $100a^2+b$ is the given ſquare Number, whoſe Root is ſuppoſed $10a+e$, therefore $\overline{10a+e}^2=100a^2+20ae+ee=100a^2+b$, whence $e=\frac{b}{20a+e}$, but $10a$ is much greater than $e$, whence $\frac{b}{20a}$, or $\frac{\frac{b}{10}}{2a}$ is nearly equal to $e$: Conſequently if $\frac{b}{10}$ be divided by $2a$, ſo that $e$ the Quotient being annexed on the right hand $2a$, and that multiplied by $e$, the Product may be equal to $b$, then is $10a+e$ the Root ſought; but if no ſuch Number ($e$) can be found, take the next leſs, and place it on the right hand $2a$ as before, the Product of which, and the Diviſor ſo found, muſt be taken from $b$, and the Remainder ſhews how much the ſaid Number exceeds an exact Square Number. From hence, and the firſt Lemma, we deduce this Rule.

*To find the ſquare Root of any given Number.*

Let the Number whoſe Root is ſought be pointed, beginning at the Units place, and ſo on towards the right hand, placing a point over every other Figure: Then ſeek among the ſimple Roots, that, whoſe Square is equal, or the next leſs, to the firſt Period, this is the firſt Figure of the Root;

let it be ſquared and taken from the firſt Period, and the next Period annexed to the Remainder, ſo have you what is called a *Reſolvend*; let this wanting the Units place (which is equivalent to $\frac{b}{10}$) be conſidered as a Dividend, and twice the firſt Figure of the Root the Diviſor, the Quotient of this Diviſion is the ſecond Figure of the Root, which being annext to the ſaid Diviſor, the Product thereof by the ſaid ſecond Figure, muſt be taken from the Reſolvend, and another Period annexed to the Remainder, will form a new *Reſolvend*; this wanting the Units Place, is again to be conſidered as a Dividend, and twice the Figures already found of the Root the Diviſor, the Quotient of which is the third Figure of the Root, which being (as before) annext to the laſt Diviſor, and the whole multiplied thereby, the Product taken from the laſt Reſolvend gives a Remainder, to which another Period muſt be annext for a new Reſolvend; and with this, and the Figures already found of the Root, you muſt proceed as before, to get the fourth Figure, and ſo on till all the Periods are brought down: But if there ſhould be ſtill a Remainder, then the given Number has no exact Root, but you may approach it at pleaſure, by annexing two Cyphers to every ſuch Remainder, for a new Reſolvend; this will be made plain by the following Example.

*Example* I. Let it be required to find the ſquare Root of 273529, this pointed, ſtands thus 273529, from whence it appears (from Lemma I.) there is three Figures in the Root (if a ſquare Number); The firſt Period is 27, which is not an exact ſquare Number, but the next leſs is 25, whoſe Root 5 is the firſt Figure of the Root ſought; then 25 taken from 27 leaves 2, to which I annex

35 (the ſecond Period) ſo we have 235 for the firſt Reſolvend, of which 23 is the Dividend, and 10 (the double of 5) the Diviſor; but 23 divided by 10, gives 2 in the Quotient, which is the ſecond Figure of the Root, and this annext to the Diviſor 10 gives 102, and 102 multiplied by 2 gives 204, which taken from the Reſolvend 235 leave 31, to which I annex 29 (the next Period) ſo we have 3129 for a new Reſolvend, of which 312 is the Dividend, and 104 (the double of 52) the Diviſor. Then 312 divided by 104 gives 3 in the Quotient, which annex'd to 104 makes 1043, this multiplied by 3 gives 3129, and this taken from 3129, the laſt Reſolvend, there remains (0): And becauſe all the Periods are brought down and no Remainder, we may conclude 523 is the Root ſought. See the following Work, where obſerve $a$ denotes the firſt Figure of the Root, $a'$ the two firſt, $a''$ the three firſt, &c. alſo $e$ denotes the ſecond Figure of the Root, $e'$ the third Figure, $e''$ the fourth Figures, &c. R is the firſt Reſolvend, R$'$ the ſecond, R$''$ the third, &c. ſo that $a'=10a+e$ $a''=10a'+e'$, $a'''=10''+e''$, &c. But when any of the $e$'s is a Decimal, then the next $a$ is only equal to the Sum of this $e$, and the laſt $a$. For in that Caſe $\overline{a+e}^2=a^2+b=a^2+2ae+ee$, or $e=\sqrt{\frac{b}{2a+e}}$ and $a'=a+e$, which will be manifeſt from the following Example.



2735 29

$$\begin{array}{r} 273529\ (5=a \\ 25 \\ \hline \left.\begin{array}{r} 2a=10 \\ 20a+e=102 \end{array}\right\} 235=R\ (2=e\ \therefore a'=52 \\ 204=e\times 20a+e \\ \hline \left.\begin{array}{r} 2a'=104 \\ 20a'+e'=1043 \end{array}\right\} 3129=R'\ (3=e'\ \therefore a''=523 \\ 3129=e'\times 20a'+e' \\ \hline 0000 \end{array}$$

*Examp.* II. What is the ſquare Root of 4786954; this being pointed ſtand thus 4786954, ſo that by Lemma Iſt, there will be four Figures in the integral Part of the Root, the Operation as follows.

4786954

$$\left.\begin{array}{l} 2a=4 \\ 2a+c=41 \end{array}\right\} \begin{array}{l} 4=a^{2} \\ 78=R \end{array} \quad (1=c \therefore 10a+c=21=a' \qquad 47869.54\ (2=a$$

$$\left.\begin{array}{l} 2a'=42 \\ 2a'+c'=428 \end{array}\right\} \begin{array}{l} 41=\overline{2a+c \times c} \\ 3769=R' \end{array} \quad (8=c' \therefore 10a'+c'=218=a''$$

$$\left.\begin{array}{l} 2a''=436 \\ 2a''+c=4367 \end{array}\right\} \begin{array}{l} 3424=\overline{2a'+c' \times c'} \\ 34554=R'' \end{array} \quad (7=c'' \therefore 10a''+c''=2187=a'''$$

$$\left.\begin{array}{l} 2a'''=4374 \\ 2a'''+c'''=4374,9 \end{array}\right\} \begin{array}{l} 30569=\overline{2a''+c'' \times c''} \\ 398,500=R''' \end{array} \quad (,9=c''' \therefore a'''+c'''=2187,9=a''''$$

$$\left.\begin{array}{l} 2a'''=43758 \\ 2a'''+c'''=43758,1 \end{array}\right\} \begin{array}{l} 393741=\overline{2a'''+c''' \times c'''} \\ 475900=R'''' \end{array} \quad (,01=c'''' \therefore a''''+c''''=2187,91=a$$

3,8319 the Remainder.

And if to this Remainder two Cyphers were added, and $a^{[1]}$ doubled as before, we might proceed to find another Figure, and ſo another, till as many Figures were had as would be ſufficient; but after the Root is found pretty near, a good many of the remaining Figures will be had by dividing by the laſt Diviſor; thus in the preſent

if 3,8319 be divided by 4375,8 the Quotient, 0,00876 are the Figures to be annext to 2187,91; ſo that the ſquare Root of 4786954 is 2187,910876 nearly, which was required: Or having obtained the Root pretty near, you may treble the Figures of the Root already found by this Rule; let Q be the Number whoſe Root is ſought, and $r$ denote any of the above Remainders which we expreſſed by R, R′, R″, R‴, &c. with remaining Figures annext; alſo put $q$ for the Figures already found of the Root, when the Remainder $r$ was got, then the Figures ſtill wanting of the Root will be expreſs'd by $\frac{2rq}{4q^2+r}$, and in this if $r$ be ſo ſmall in reſpect of $4q^2$ that it may be neglected, then this Expreſſion becomes $\frac{r}{2q}$, the ſame with what we have above in Words. But to ſave the trouble of ſquaring $2q$ in order to get $4q^2$, you need only ſubſtract $r$ from Q, and the Remainder is $q^2$. The Reaſon of this is omitted, as being eaſily deduced from Dr. *Hally's Rational Theorem for Pure Powers*, wherein he ſhews if $b$ denotes how much any Number exceeds the $m$th Power of $a$, that the $m$th Root thereof ſhall exceed $a$ by $\frac{ab}{ma^m+\frac{m-1}{2}b}$, or that $\sqrt[m]{a^m+b} = n + \frac{ab}{ma^m+\frac{m-1}{2}b}$ *proximè*.

If the Number given was a decimal Fraction, or compoſed of Integers and Decimals, then the Units place being found, let the Integers (if any) be pointed as before, and for the Decimals a point muſt be put over every other Figure towards the right hand, beginning at the place of hundreds, or ſeconds; after which the Method of Operation is

the

the ſame as if the whole were Integers, only remember to ſeparate as many Figures for Decimals in the Root, as are the Points over the Decimal Parts of the Number it ſelf. This will be plain from an Example. Let it be required to find the Square-Root of 4226,754695, this pointed ſtand thus $\dot{4}\dot{2}\dot{2}\dot{6},\dot{7}\dot{5}\dot{4}\dot{6}\dot{9}\dot{5}$, from whence it appears there will be two integral, and three Decimal Figures in the Root. The Operation follows.

$4226,754695\ (65 = a$
$36 = a^2$

$2a = 12$
$2a + e = 125$ } $626 = R'\ (5 = e \therefore 10a + e = 65 = a'$
$625 = 2a + e \times e$

$2a' = 130,$
$2a' + e' = 130,0$ } $175 = R'\ (,0 = e' \therefore a' + e' = 65,0 = a''$
$000 = 2a' + e' \times e'$

$2a'' = 130,0$
$2a'' + e'' = 130,01$ } $17546 = R''\ (,01 = e'' \therefore 00a'' + e'' = 65,01 = a'''$
$13001 = 2a'' + e'' \times e''$

$2a''' = 130,02$
$2a''' + e''' = 130,023$ } $454595 = R'''\ (,003 = e''' \therefore a''' + e''' = 65,013 = a''''$
$390069 = 2a''' + e''' \times e'''$

$,064526 = \text{Remainder.}$

And

And here if the Remainder ,064426 be divided by 130,02, the laſt Diviſor, the Quotient ,0004955 is nearly the remaining Part of the Root, which therefore is 65,0134955: but we ſhall take this Opportunity to illuſtrate what was ſaid of tripling the Figures already found of the Root; in this Example $Q=4226,754695$, and having got two Figures of the Root, (*viz.* 65) we find $R'=,175$, to which add the other Figures, and we have $r=1,754695$, whence $Q-r=q^2=4225$, as is evident, (for $q=65$) hence $4q^2=16900$, therefore $4q^2+r=16901,754695$, and $2rq=130\times1,754695$ $=228,11035$, therefore $\frac{228,110350}{16901.754695}$ is the remaining part of the Root nearly; but in this Diviſion we need only fix Figures of each, the Operation is is as hereunder,

```
1|6|9|0|1|8) 228,110 (,013496
             169018
             ------
              59092
              50705
              -----
               8387
               6761
               ----
               1626
               1521
               ----
                105
                ---
                102
                ---
                  3
```

CHAP.

## CHAP. III.

### *The Conſtruction of the Sliding-Rule, with the Uſe thereof, and the Reaſon of the Method of Operation thereon Demonſtrated.*

SINCE, as we obſerved before, the Practice of Gauging almoſt entirely depends on the Knowledge of the Sliding-Rule, it muſt be of great Importance to the Gentlemen of the Exciſe to be acquainted, not only with the Method of Operation thereon, but the Reaſon thereof: 'Tis true indeed, ſo many have already wrote on this Subject, that it will ſeem ſuperfluous to attempt any thing of this kind at preſent; but beſides obſerving that thoſe Authors, (eſpecially the moſt modern) not being ſufficiently acquainted with the Theory, have chiefly confined themſelves to the Practice; ſo what they have done even in that is ſo irregular, that I fear few of their Readers have a ſufficient acquaintance with the Nature and Properties of this valuable Inſtrument. Therefore in what follows I deſign to trace it from its Original, and that in ſo regular and plain a Manner, as to render the Conſtruction and Uſe thereof ſo eaſy, as to come within the Power of thoſe but indifferently verſed in Mathematical Knowledge: And in order to this, in the firſt place, the Nature and Properties of Logarithms muſt be firſt explained, becauſe the principal Lines on the Sliding-Rule are logarithmical, for which purpoſe we ſhall lay down the following Doctrine.

#### SECT. I. *Of the Properties of Powers and Exponents.*

In Specious Computations, every one is at Liberty to chuſe after what manner he will ſignify the

the Sum, Difference, Product or Quotient of any Quantities *a*, *b*, *c*, *d*, &c. for it is a matter of Indifference by what ſigns or Manners they are repreſented, ſo that we are conſtant in our Practice, and always follow the ſame Rule; for thereby we ſhall ever be able to go back from any Sum or Product to its component Factors, which is ſufficient in all kinds of computation: It is common to expreſs the Sum of the above Quantities thus, $a+b+c+d$ &c. and their Product thus, $axbxcxd$ &c. or for brevity ſake, ſome omit the mark (x), and ſignify their Product thus, *abcd*, by joining the ſeveral Factors together, as in forming Words from Letters. From this laſt way of expreſſing a Product, if the ſeveral Factors were the ſame, (viz. *a*), then thoſe Products would be thus *aaaa*, &c. which are called the Powers of the Quantity *a*; the firſt being *a*, the ſecond *aa*, the third *aaa*; that is any Power of *a*, expreſſed after this manner, is ſignify'd by as many *a*'s ſet together, as is the Number expreſſing that Power. But it is found troubleſome to expreſs large Powers of Quantities after this manner, hence ſome who had no occaſion to conſider any Powers beyond the 5th or ſurſolid, choſe to denote the ſeveral inferiour Powers by a particular Letter or Letters adjoined to each, as the Square of A, was denoted by A*q*; the Cube of A, by A.*c*; the fourth Power of A, by A.*qq*; the fifth Power of A, by A.*qc*, and ſo on: but this, as obſerved already, is only fitted for ſmall Powers, ſo that the firſt Method was preferable. But the moſt illuſtrious *Newton* has introduced a new Method for this purpoſe, both more eaſy and more general than the others, extending to Roots as well as Powers, and that how high ſoever; the Properties of which we ſhall deliver in the following Propoſitions, after which the

the Reaſon of the Method of Operation by Logarithms will be eaſy.

*Definition.* Any Power of a Quantity *a*, is expreſſed by the ſaid Quantity *a*, with a Number called *the Index*) placed above towards the right hand, expreſſing the Place it has from Unity in a Geometrick Series, whoſe firſt Term is 1, and Ratio the Root it ſelf *a*, as 1, *a*, *aa*, *aaa*, *aaaa*, *aaaaa*, *&c.* thus for Example, *aaaa*, being the 4th Term from 1, is expreſſed by $a^4$; alſo *aaaaa*, being the 5th Term from 1, is expreſſed by $a^5$, and ſo of others.

PROP. I.

*The Index of the Product of any* 2 *Powers of the ſame Quantity, is equal to the Sum of the two Indexes of the Factors*, viz. $a^m \times a^n = a^{m+n}$.

The Indexes both here and in what follows, are ſuppoſed Integers.

For by what was ſaid above $a^m = aaaaa$, *&c.* till the Number of *a*'s is *m*, and $a^n = aaaaa$, *&c.* till the Number of *a*'s is *n*; whence $a^m \times a^n = aaaaaaaaaa$, *&c.* till the Number of *a*'s is $m+n$, which Power of *a* (by the Definition) is expreſſed $a^{m+n}$. *Q. E. O.*

PROP. II.

*The Index of the* m*th Power of* a, *viz.* $a^m$, *raiſed to the* n*th Power, viz.* $\overline{a^m}^n$, *is equal to* $m \times n$.

For, by the Definition, $\overline{a^m}^n = a^m \times a^m \times a^m \times a^m$ *&c.* till the Number of Factors is *n*, but $a^m \times a^m \times a^m \times a^m$ *&c.* is, by the laſt, equal to $a^{m+m+m+m}$ *&c.* and the Number of *m*'s being *n*, 'tis evident $m+m+m+m$ *&c.* $= m \times n$, whence $\overline{a^m}^n = a^{m \times n}$. *Q. E. O.*

PROP.

PROP. III.

*The Index of the Quotient of one Power of any Quantity, viz.* $a^m$, *divided by another Power of the ſame Quantity, viz.* $a^n$, *is equal to the Index of the Diviſor* n, *taken from that of the Dividend* m; *that is* $\frac{a^m}{a^n} = a^{m-n}$.

For put $\frac{a^m}{a^n} = a^x$, and let the Equation be multiplied by $a^n$, then, by *Propoſit.* I. we have $a^m = a^{n+x}$; therefore, ſince the Roots are the ſame and the Powers equal the Indexes muſt be ſo too, whence $m = n+x$, or $x = m-n$. Q. E. O.

*Cor.* I. From this Propoſition it is plain $a^0 = 1$ for if $m = n$, then $x = 0$, but $\frac{a^m}{a^m} = 1$, therefore $a^x = a^0 = 1$.

*Cor.* II. Alſo from hence it follows $\overline{a^n = a}^{-n}$; for $\frac{a^m}{a^n} = a^{m-n}$, hence if $m = 0$ then $\frac{a^0}{a^n} = a^{-n}$ but $a^0 = 1$, thence $\overline{a^n}^{1} = a^{-n}$.

SECT. II. *Tranſition to Logarithms.*

From theſe Propoſitions we are enabled to demonſtrate the Method of Operation by Logarithms: For 'tis moſt evident, if between 1 and any Number (ſuppoſe 10,) an infinite, or only a very great Number of Proportionals 1000000 be taken, thereby forming a Geometrical Series, ſome of them

them muſt approach very near to the natural Numbers, 2, 3, 4, 5, 6, 7, 8, 9; and by continuing this Series, the ſubſequent Terms will alſo approach very near the following Numbers 11, 12, 13, 14, *&c.* till the whole Scale of Numbers to any deſired one, (as 100000 the Extent of the largeſt Table of Logarithms) is compleat: Hence, thro' the natural Numbers themſelves do not form a geometrick Series, yet on the above Suppoſition, they will obtain Places, very near, in ſuch a Series; and then,

Definition. *The Place from* 1, *which any of them has in that Series, which is ever the Index of the ſecond Term, is called it Logarithms.*

Thence it follows that Logarithms may be ſaid to be the *Exponents* of Ratio's.

Having given this Idea of the Nature of Logarithms, we ſhould proceed to ſhew how they might be computed: But ſince every body at preſent may be ſuppoſed furniſh'd with Tables, we ſhall from thence take ſuch as will enable us to find the Diviſions on the Sliding-Rule: But that nothing might be omitted to render this Treatiſe compleat, we ſhall add, at the End, Rules for computing them, as delivered by the incomparable Dr. *Halley*, with ſome others, ready in Practice.

Now let $a$ denote the firſt Term of the infinite Scale of Proportionals, which is taken betwixt 1 and 10; and let us denote the Logarithm of any Number by the Number it ſelf with L: prefixt: Thus the Logarithm of 2 we expreſs by L:2; the Logarithm of 3 by L:3; the Logarithm of 4 by L:4, *&c.* then 'tis manifeſt this infinite Scale of Proportionals will ſtand thus, $1, a, a^2, a^3, a^4,$ $\ldots\ldots 2=a^{L:2} \ldots 3=a^{L:3} \ldots 4=a^{L:4}$ $\ldots$ *&c.* $10=a^{100000}$, *&c.*

PROP.

### PROP. I.

*The Sum of the Logarithms of any Numbers, is equal to the Logarithms of the Product of thoſe Numbers.*

For let the two Numbers be denoted by A and B, then I ſay, $L:A+L:B=L:A\times B$.

By what was juſt obſerved it is plain, $A=a^{L:A}$, and $B=a^{L:B}$, therefore $A\times B=a^{L:A+L:B}$ (*per* Prop. I. *of the laſt Section*,) but (*per Definition*) the place which any Number obtains in the Series is its Logarithm, and that is ever the Index of *a* the ſecond Term; whence $L:A+L:B=L:A\times B$. *Q. E. O.*

### PROP. II.

*The Logarithm of the Quotient of any two Numbers, is equal to the Logarithm of the Diviſor, taken from that of the Dividend,*

*that is,* $L:\frac{A}{B}=L:A-L:B$.

For if the above Value of A be divided by that of B, we have $\frac{A}{B}=a^{L:A-L:B}$, by *Propoſit.* III. foregoing. Whence by what was already obſerved, $L:\frac{A}{B}=L:A-L:B$. *Q. E. O.*

### PROP. III.

*The Logarithm of any Power of a Quantity is equal to the Logarithm of the Quantity it ſelf multiplied by its Index, that is* $L:A^m=m\times L:A$.

For ſince $A=a^{L:A}$, 'tis manifeſt, (*by* Prop. II. *of the laſt Section*) $A^m=a^{m\times L:A}$, whence 'tis evident $m\times L:A=L:A^m$. *Q. E. O.*

*Cor.*

*Cor.* From this Propoſition it appears, if there are two Scales of Logarithms, in which the Logarithm of any Number in the one is double of the Log. of the ſame Number in the other, then the Logarithms of all Numbers taken from the former, ſhall be the Logarithms of the Square of the ſame Numbers taken from the latter: For we have ſhewn that $L : A^m = m \times L : A$, therefore if $m = 2$ then $L : A^2 = 2 \times L : A$. Again, if there are two Scales, or ſorts of Logarithms, where the Logarithm of every Number in the one is triple the Logarithm of the ſame Number in the other, then the Logarithms of all Numbers in the former ſhall be the Logarithms of the Cubes of the ſame Numbers taken from the latter: For in the laſt by making $m = 3$ we have $L : A^3 = 3 \times L : A$, and A is put indefinitely for any Number. Theſe Corollaries muſt be well conſidered, becauſe on them depends the Reaſon of the Operation of a ſingle Radius (of the Rule) ſliding againſt a double or triple one, by which we find a fourth Proportional to theſe three Quantities, $Q^2 : R^2 : S$: or to theſe three, $Q^3 : R^3 :: S$: when the Value of Q, R and S are only given. Having now demonſtrated the method of the Operation by Logarithms, which was never generally done before, we ſhall proceed to the Conſtruction of the Sliding-Rule.

## SECT. III. *The Conſtruction of the Sliding-Rule.*

The Rule we are to conſtruct, is in the Form of a Parallelopipedon, on three Faces of which are three graduated Sliders, ſliding by ſix graduated Lines on the Rule; the firſt Face is repreſented by Fig. 1. (*Plate* 1.) the ſecond is repreſented by

Fig. 2. the third by Fig. 3. and the fourth by Fig. 4. of each of theſe in their order as they are marked A, B, C, D, E, MD, SL, SS, and N. The Lines graduated on the firſt Side, are four Lines of Numbers, or Logarithms, or rather three, becauſe the Sliding-piece marked B has two Lines, each divided in the ſame manner: A is the firſt of theſe Lines; to delineate which, let the Sector be opened till 10, 10, on the Line of Lines be equal to the Diſtance betwixt one and ten on A; then ſince the Logarithms of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, are 0, 301, 477, 602, 699, 778, 845, 703, 954, 1,000 reſpectively from the Line of Lines on the Sector thus opened, take theſe Numbers, and lay them reſpectively from 1 along the ſame Line A, ſo ſhall you have the Points anſwering the Numbers, whereof the ſaid Diſtances are Logarithms; the ſame Diſtances laid from 10, (which on the Rule is only marked 1) will give the other Diviſions on this Line, it containing its Radius twice; then ſince the firſt of theſe Diſtances, (*viz.* betwixt 1 and 2) is divided into 50 Parts, let the Logarithms of $1.\frac{1}{50}$, $1.\frac{2}{50}$, $1.\frac{3}{50}$, $1.\frac{4}{50}$, *&c.* be taken from the Table of Logarithms, and the diſtance anſwering thoſe Log. taken from the Sector opened as before, and laid from 1 along A; ſo ſhall you have all the Diviſions of the Space 1, 2. Again the Space 2,3; 3,4; 4,5 are each divided into 20 Parts; wherefore, let the Logarithms of $2.\frac{1}{20}$, $2.\frac{2}{20}$, $2.\frac{3}{40}$, $2.\frac{4}{40}$, *&c.* be alſo taken from the Sector open'd as before, and laid from 1 along A, ſo will you have all the Diviſions to 3, then the Logarithms of 3, $3\frac{1}{20}$, $3\frac{2}{30}$, $3\frac{3}{20}$, *&c.* to 4; then the Log. of 4; $4.\frac{1}{20}$, $4.\frac{2}{40}$, $4.\frac{3}{20}$, $4,\frac{4}{20}$, *&c.* to 5, and the diſtances correſponding to theſe ſeveral Logarithms, being ſucceſſively taken from the Sector opened as before, and laid from 1 towards A, will give you all the Diviſions betwixt 2 and 5 in both

Ra-

Radius's. Then ſince the remaining Spaces, *viz.* 5,6, and 6,7, and 7,8, and 8,9, and 9,10, are each only divided into 10 Parts; let the Logarithms of $5\frac{1}{10}$, $5\frac{2}{10}$, $5\frac{3}{10}$, *&c.* to 6 be found; alſo thoſe of $6\frac{6}{10}$, $6\frac{2}{10}$, $6\frac{3}{10}$, *&c.* to 7, and ſo on till you have found all the Logarithms of the Series, $5\frac{1}{10}$, $5\frac{2}{10}$, $5\frac{3}{10}$ . . . . 6, $6\frac{1}{10}$, $6\frac{2}{10}$, $6\frac{3}{10}$, . . . $7\frac{1}{10}$, $7\frac{2}{10}$, $7\frac{3}{10}$, . . . $8\frac{1}{10}$, $8\frac{2}{10}$, $8\frac{3}{10}$, . . . $9\frac{1}{10}$, $9\frac{2}{10}$, $9\frac{3}{10}$, . . . . 10; then let the Diſtance correſponding to each of theſe Logarithms be taken from the Sector (ſtill open'd as before,) and laid from 1 along A, ſo will you have all the Diviſions on this Line.

The Lines B, C, are exactly the ſame with A, ſo nothing further is neceſſary to be ſaid on them, but only inſtead of A in the above Conſtruction, to read B or C; the Line D has a ſingle Radius exactly double to A, B, *&c.* ſo that the Diſtance from 1 correſponding to every Number on this, is juſt double of the Diſtance from 1 correſponding to the ſame Number on any of them.

The Line E has a triple Radius, each of which is a third Part of the Radius D, ſo that the Diſtance from 1 correſponding to any Number on this Line is exactly $\frac{1}{3}$ of the Diſtance from 1 on the ſame Number on D, or two thirds of the Diſtance betwixt one and the ſame Number on any of the Lines A, B, and C; whence it is manifeſt from the Diviſions on A, all thoſe on B, C, D and E are immediately had. We are now come to the Line mark'd MD, which is alſo Logarithmetic, and the ſame with A, B, C, D, but inverted, and does not begin from 1 as the others do, but from 21.5042, and ſo on in an inverted Order, ſo that the Order of the Numbers on this Line are 2 (or 200) 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 9, 8, 7, 6, 5, 4, 3, and the laſt Diviſion is 2,15042, by

which dividing 215, you have 100 for the Quotient, which is the ſame with the laſt Diviſion on A, B, C, and D; by which means it will be found that the Point 1 will be exactly over againſt the Points on A, marked with M B, and a Braſs Pin, which correſponds with the Number 2150,42, or 215,042, or 21,5042: the firſt of theſe being the cubic Inches in a Buſhel of Malt, for if from the Logarithm of 10, or 100, the Logarithm of 2,15042, or 21,5042 be taken, the Remainder ſhews the Diſtance of *one* from the end of the Line MD, which is exactly the ſame with the Diſtance of the Points MB from the end of the Line A, that being alſo equal to the Difference betwixt the Logarithm of 10 or 100, and 2,15042 or 21,5042. The Reaſon why this Line is inverted, and its beginning and ending as above, will be ſhewn when we come to the Uſe of the Sliding-Rule.

*Cor.* From what has been ſaid, it is plain the Sliding-Rule is analogous to a Table of Logarithms, the Diſtances of the Diviſions on A, B, C, D and E, being as the Log. of their Ratios.

Having now ſhewn how the Lines on two oppoſite Faces of the Sliding-Rule are obtained, we come to the other two; on one of which are four Lines, *viz.* a Slider with two Lines, which may be reduced to one, they being only the ſame Diviſions, continued to its two Edges, and two others on the Rule it ſelf, to operate with the Slider; the Slider itſelf mark'd N, is exactly the ſame with the Lines A, B, C, already deſcribed, ſo that what was ſaid of them is ſufficient for this. One of the fix'd Lines on this ſide is mark'd with S: L: or *Seq. ly.* whoſe office, with the aſſiſtance of the Slider, is to find the Quantity of Liquor drawn, or remaining in a Caſk not full, and whoſe

whoſe Axis is parallel to the Horizon : The other fixed Line is mark'd SS, or *Seq. St.* whoſe Uſe is to find the Quantities of Liquor drawn out of a Caſk when its Axis is perpendicular to the Horizon : But the Conſtructions of theſe Lines depending on Principles not yet laid down, we ſhall *poſtpone* them till we come to *Ullaging*, which is handled in the laſt Part.

On the other Side oppoſite this are 3 fixed Lines, whereon is wrote *Spheroid Variety*, *&c.* whoſe Uſes are to find what is called *the mean Diameter*, orto reduce a Caſk of any of the three Varieties to Cylinders; but that being of little uſe, and ſo eaſily done without a mean Diameter, it is not worth ſpending any Time on their Explication. On the Inſides of the two firſt Sliders are placed two Lines, the the one numbred from 13 to 16, and the other from ,42 of a Gallon to 3,61 Gallons; which is only a Table to ſhew the Contents in Ale Gallons of all Cylinders, whoſe Depths are 1 Inch, and Diameter from 13 Inches to 36 incluſive, which are eaſily computed from the Rule following for gauging Cylinders, ſo that we ſhall not dwell any longer on them, but proceed to

## SECT. IV. *The Uſe of the Sliding-Rule.*

*And firſt, of Numeration, or the Method of eſtimating the Values of the Diviſions on the Sliding-Rule, for which this is a general Rule :* For the Lines A, B, C, D, MD and N; the Value of the firſt on the Rule may be aſſign'd at pleaſure, either 1, 10, 100, 1000, *&c.* or ,1 ; ,01 ; 001, *&c.* and whatever that is, the firſt 2 is twice as much, the firſt 3 thrice as much, the firſt 4 four times as much, and ſo on to the ſecond 1, which is ten times the firſt; the ſecond 2, 3, 4, *&c.* being ten times the Value of the firſt 2, 3, 4, 5, 6, *&c.* And

where there are three 1's, or a triple Radius, the third 1, 2, 3, 4, 5, 6, *&c.* are 100 times the Value of the firſt, or ten times thoſe of the ſecond; and this may ſerve for the integral Diviſions; but thoſe being known, the intermediate ones are eaſy; for 'tis but conſidering into how many Parts the Space betwixt any two of the integral ones is divided, and the difference of their Values, for then the firſt of them is eſtimated by a Fraction, whoſe Numerator is the Difference of the Values of the adjacent Figures on each ſide, and its Denominator is the Number of Parts into which the Space betwixt the Integers is divided; the ſecond Diviſion is twice the Value of the firſt, the third thrice its Value, *&c.* and if to any of theſe be added the Value of the preceding Figure, you will have the Number to which that Diviſion on the Rule correſponds. We will illuſtrate this by an Example or two on the Lines A,E. Let the firſt 1 at the end of the Rule on A ſtand for 1, then the ſecond 2 ſtands for 20; now ſince the Space betwixt the ſecond 2 and 3 is divided into 20 Parts, this is the Denominator of all the Diviſions betwixt the ſecond 2 and 3, and the Difference betwixt the ſecond 2 and 3, *viz.* 30—20 is 10; ſo from the Rule given above, the firſt Diviſion after the ſecond 2, is $20\frac{10}{20}=20\frac{1}{2}$; the ſecond Diviſion $20+\frac{2\times 10}{20}=21$; the third $20+\frac{3\times 10}{20}=21\frac{1}{2}$, *&c.*

Let us take a ſecond Example on the Line E, which has a triple Radius, and let the firſt 1 ſtand for 10, then the third 1, (which often is marked 10) ſignifies 1000; and the third 5 ſtands for 5000; then becauſe the Space betwixt this 5 and the following 6 is divided into ten Parts, and the Difference of their Diviſions is 1000; the firſt Diviſion after the third 5, ſignifies $5000+\frac{1000}{10}=5000+100=5100$, the ſecond is $5000+2\times\frac{1000}{10}=5000+200=5200$, and ſo of the reſt.

*Multi-*

## *Multiplication by the Sliding-Rule.*

### PROP. I.

*Two Numbers being given, it's required to find their Product by help of the Sliding-Rule.*

*Rule.* Set 1 on B to either of the Factors on A, then againſt the other Factor on B is the Product on A.

For the two Numbers being repreſented *a*, *b*, and their Product by *p*, we have $a \times b = p$, or (*by Prop.* I. *of the* 2*d Sect.*) $L:p = L:a + L:b$; now 'tis plain, when 1 on B is removed to correſpond with *a* on A, then the Diviſion on A, which is oppoſite to *b* on B, ſhall be diſtant from 1 on A, by the Sum of the Diſtances belonging *a*, *b*; but thoſe Diſtances are Logarithms of the Numbers *a*, *b*, by what hath been ſaid above; whence the Diſtances of the Diviſion on A, which is oppoſite *b*, on B ſhall be expreſſed by $L:a + L:b$; but the $L:a + L:b = L:p$ from the foregoing, whence the Number anſwering that Diviſion is equal to *p*, the Product of *a*, *b*. Q. E. O.

*Example* I. Let the Product of 15 and 12 be ſought. By the above Direction, ſet 1 on B to 12 on A, then againſt 15 on B, I find (by the preceeding Numeration) is the Number 1,80, the Product ſought.

There may occur ſome Difficulties to Beginners in theſe Matters, for it may happen either that after 1 on B is placed to one of the Factors on A, the other Factor may not be found on B, becauſe of the ſhortneſs of the Rule, or of its not having a ſufficient Number of Radius's, or that when the ſecond Factor is found on B, it falls beyond the Line A; in both which Caſes you muſt divide

one or both the Terms by 10, 100, 1000, *&c.* not only till they are both to be found on their reſpective Lines, but alſo that the Diviſion corresponding with the Number on B, may fall oppoſite ſome Diviſion on A: and then you are to multiply the Product thence ariſing, by what both the Factors were divided by. The Reaſon of this is manifeſt; for the Factors being *a*, *b*, and their Product P, *viz.* $a \times b = p$, then $\frac{a \times b}{n} = \frac{P}{n}$, conſequently $\frac{p}{n} \times n = p$. We will illuſtrate this by Examples.

*Example* I. Let the Product of 3 and 245 be required, 'tis plain the firſt 1 on B being eſteemed 1, the laſt Diviſion on that Line is ,100, and therefore 245 is off the Rule; but let 245 be divided by 10, and the Factor become 3 and 24,5, which by the Rule gives 73,5 for the Product; therefore becauſe 245 was divided by 10, I multiply 73,5 by 10, and get 735 for the Product of 3 by 245.

*Example* II. Let the two Numbers be 400 and 350, here taking the firſt 1 on B for *one* 350 is off the Rule, ſo I divide it by 10, and have 35; but when 1 on B is ſet to 400 on A, (the firſt 1 on that Line being called 100) then 35 falls beyond the Line A, ſo I divide it again by 10, that is 350 is divided by 100, and againſt the Quotient (3,5) on B is 1400; but ſeeing 350 was divided by 100, I multiply 1400 thereby, and have 140000 for the Product of 350, by 400.

*Diviſion*

## *Diviſion by the Sliding-Rule.*

### PROP. II.

*Two Numbers being given, 'tis required to find their Quotient by help of the Sliding-Rule.*

*Rule.* Set 1 on A to the Diviſor on B, then oppoſite the Dividend on B, is the Quotient on A.

The two Numbers being as above called $a, b$, of which $b$ is the Dividend, and $a$ Diviſor, and the Quotient $q$, then $\frac{d}{a}=q$, or L: $b$—L: $a$=L: $q$ (*by Prop.* 2. *Sect.* 2.)

But when 1 on A is ſet to $a$ on B, 'tis evident the Number on A oppoſite $b$ on B, ſhall correſpond to the Exceſs by which the Diſtance from 1 to $b$, exceeds that from 1 to $a$: and theſe Diſtances being Logarithms of thoſe Numbers, it follows that the Diſtance from 1 on A oppoſite $b$ on B, ſhall be expreſs'd by L: $b$—L: $a$, whence *per Prop.* 2. *Sect.* 2. the Number belonging to that Diſtance is $\frac{b}{a}=q$. Q. E. O.

*Examp.* Let the Diviſor be 6, and dividend 96. According to the Rule I ſet 1 on A to 6 on B, then oppoſite 96 on B is 16 on A, which is the Quotient ſought.

But here as well as in the preceeding Propoſitions, ſome ſeeming Difficulties analogous to thoſe there mentioned may occur, but the ſame Remedy is ſufficient. For let both the Terms be divided by any Power of 10, as 10, 100, 1000, *&c.* and the Quotient thence ariſing is to be multiplied by a Fraction, whoſe Numerator is the Diviſor of the Dividend, and Denominator that of the Diviſor; but both here and in the preceeding Propoſition, 'tis ſcarce ever neceſſary that any more than one

one of the Factors, or Terms, need be divided; the Reaſon of this is evident, for ſince $\frac{b}{a}=q$; if $m$ and $n$ are any Powers of 10, then $\frac{a}{n})\frac{b}{m}(=\frac{nb}{ma}$, therefore $\frac{na}{mb}\times\frac{m}{n}=\frac{a}{d}=q$.

One Example will be ſufficient to illuſtrate this: Let it be required to find the Quotient ariſing from 1944, being divided by 54; then calling the firſt 1 on *A one*, and the firſt 1 on B *ten*, and ſetting one on A to 54 on B, the other Term 1944 is not on B, but dividing by 10, the Quotient 194,4 on B is againſt 3,6 on A; this multipled by $\frac{10}{1}$ (for the above Reaſon) and I find 36 for the true Quotient.

## *The Rule of Three by the Sliding-Rule.*

### Prop. III.

*Any three Numbers being given, 'tis required to find a fourth, which ſhall be to the third, as the ſecond is to the firſt.*

*Solution.* Set the firſt on B, to the ſecond on A, then oppoſite the third on B, is the fourth on A.

For let us denote the three Numbers by $a$, $b$, $c$, and that ſought by $x$, then $a:b::c:x$; therefore (*per* 16. VI*th* Euclid) $x=\frac{b\times c}{a}$, and (*per Prop.* 1. *and* 3 *Sect.* 2.) $L:x=L:b-L:a+L:c$.

Now when $a$ on B is ſet to $b$ on A, then 1 on B is diſtant from 1 on A by the Difference of the Spaces agreeing to $a$, $b$, that is by the Difference of their Logarithms; hence the 1 on B is diſtant from the ſame on A, by a Space expreſſed by $L:b-L:a$: and therefore the third Term on B is diſtant from 1 on A, by its Log. added to the former

former diſtance, *viz.* by L; $b$—L : $a$+L : $c$; but L : $b$—L : $a$+L : $c$=L : $x$, hence the Number on A oppoſite the third Term on B is $x=\frac{b\times c}{a}$. Q. E. O.

Here if when the firſt on B is put to the ſecond on A, the third cannot be found on B, or that being found, it happens beyond the Line A; any, or all the three Terms may be divided by any Power of ten, till all the four Terms happen on the Rule: then the Term thence ariſing is to be multiplied by a Fraction, whoſe Numerator is the Product of the Diviſors of the ſecond and third Terms, and Denominator the Diviſor of the firſt Term; for put $m$, $n$, $r$, to denote 10, 100, 1000, *&c.* the reſpective Diviſors of the firſt, ſecond, and third Terms, then $\frac{a}{m} : \frac{b}{n} :: \frac{c}{r} : \frac{b\times c\times m}{a\times r\times n}$, but this multiplied by $\frac{n\times r}{m}$ gives $\frac{b\times c}{a}=x$, the ſame as was found before the ſaid Diviſion: And theſe are the Propoſitions whoſe Solutions are had by help of the Lines A, B, that is by two Lines of Numbers of equal Radii, or made from the ſame Scale of equal Parts. We now proceed to conſider thoſe which are had by a ſingle Radius, ſliding againſt a double or triple one, or ſuch where one is made from a Scale either double or triple to that from whence the firſt was made.

## *Of extracting the Square Root, by the Sliding-Rule.*

### PROP. IV.

*Any Number being given, 'tis required to find its Square Root by the Sliding Rule.*

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The firſt 1 on *c* ſtanding againſt the firſt 1 on D, againſt the given Number on *c* is its Root on D.

For ſince the Radius on D is double that on C, 'tis evident the Diſtance of 1 from any Number on D, is double the Diſtance of 1 and the ſame Number on C; therefore in this Poſition of the Rule, againſt any Number on D, is its Square on C, by *Cor. Prop.* 3. *Sect.* 2. whence on the contrary againſt any Number on *c* is its Square Root on D Q. E. D.

But if when the firſt 1's on each Rule are one, the Number given cannot be found on *c*, let it be divided by $\overbrace{10}^{2}$ or $\overbrace{10}^{4}$ or $\overbrace{10}^{6}$ *&c.* till it can be found on *c*; then if the Number oppoſite this Quotient on D, is multiplied by the Square Root of the ſaid Diviſor, it will be the Root ſought.

For let Q be the Number given, and *q* its Root, then $q^2=Q$, but when Q is divided by any Number $n^2$, we have $\frac{Q}{n^2}=\frac{q^2}{n^2}$ and $\frac{q}{n}=\frac{\sqrt{Q}}{n^2}$, whence if this Root is multiplied by *n*, the Square Root of $n^2$ (the Diviſor of Q), you'll have *q*, which by ſuppoſition is the Root of Q. *Q. E. O.*

*Example.* Let it be required to find the Square Root of 15129, the firſt 1's on C, D, being *one*, 'tis plain the greateſt Number on C is 100, and on D 10, therefore I divide 15129 by $\overbrace{n}^{4}=10000$, and the Quotient is 1,5129; againſt which on *c* is 1,23 on D, but becauſe 15139 was divided by $\overbrace{n}^{4}$, I multiply 123 by $\overbrace{10}^{2}=100$, and that gives 123 for the Square Root of 15129.

### Prop. V.

*Any three Numbers being given, let it be required to find a fourth, which ſhall be to the third as the Square of the ſecond is to the Square of the firſt.*

*Solution.*

*Solution.* Set the firſt Root on D to the third Term on C, then on C oppoſite the ſecond Root on D, is the Term required.

For let the three Terms given be denoted by $a$, $b$, $c$, and $x$ the Term ſought, then $a^2 : b^2 :: c : x$ by the Condition of the Queſtion, or (by the 16th VI. *Euc.*) $x = \frac{b^2 \times c}{a^2}$, whence (from *Prop.* 1, 2, 3, of *Sect.* 2.) we have $L : x = L : 2L : b - 2L : a + L : c$.

Now when $a$ on D is ſet to $c$ on C, then $b$ on D ſtands oppoſite a Number on C which is diſtant from 1, by the ſpace belonging to the Number $c$, and the Difference betwixt the Spaces correſponding to $b$, $a$ on D: but theſe Spaces being Logarithms of the Numbers themſelves, and the Radius of D double that of C, 'tis evident the Difference of thoſe Spaces on D is expreſſed by $2L : b - 2L : a$ to the Radius of C, to which the Log. of $c$ being added, *viz.* $L : C$: we have $2L : b - 2L : c$, for the Diſtance of a Number from 1 on C to the Radius of C, oppoſite to $b$ on D. Whence the Number on C agreeing this Diſtance is $\frac{b^2}{a^2} \times c = \frac{b^2 \times c}{a^2}$, but that by what was ſaid above is equal to $x$, the Term ſought. Q. E. O.

*Example.* Let the Numbers given be 4, 5, and 18, ſo that the Term or Number ſought, is a fourth Proportional to $\overline{4}^2 : \overline{5}^2 :: 18$, by the Rule, I ſet 4 on D to 18 on C, then on C oppoſite 5 on D is $28\frac{1}{8}$, the Term or Number required.

But the chief Uſe of the Lines C, D, is in meaſuring of Cylinders, and conſequently the Globe, Spheroid, Cone, and conic Conoids, which (excepting the Hyperbolics) are in a given Ratio to their circumſcribing Cylinders. For it is demonſtrated further on, if D be the Diameter

ter of any Cylinder's Baſe, $h$ its height, and $v=$ 1284, *&c.* the Diameter of a Circle whoſe Area is Unity, then the Meaſure of its Solidity is $\frac{D^2 \times h}{v^2}$ call it S, then $\frac{D^2 \times h}{v^2} = S$, or $v^2 : D^2 :: h : S$, whence from this Prop. we have the following Rule to meaſure any Cylinder by the Lines C. D.

Rule. *Set* 1,1284 *on* D *to the Cylinder's height* C; *then oppoſite the Diameter of its Baſe on* D, *is its Content on* C.

Let the Cylinder's Height be 6 Feet, and the Diameter of its Baſe 3 Feet 6 Inches.

Then ſetting 1,1284 on D, to 6 on C; againſt 3,5 on D is 57,7 on C: which is the Number of ſolid Feet in the Cylinder given. The ſame Difficulties may occur in the uſe of theſe Lines as in thoſe above: for it may happen that after the firſt Root or Term on D, is ſet to the third Term on C, the third Term or Root cannot be found on D; or that being found, it happens off or beyond the Line C; in this caſe let the given Terms be each divided by 10, 100, 1000, *&c.* or any Multiple of theſe as occaſion requires; then let the Term found be multiplied by a Fraction, whoſe Denominator is the Square of the Diviſor of the firſt Term, and its Numerator the Product of the third Term, Diviſor by the Square of that of the ſecond, ſo ſhall you have the Number ſought.

*Example* I. Let the three Numbers given be 3, 60, 7, ſo that the Term or Number required is a fourth Proportional to $\overline{3}^2 : \overline{60}^2 :: 7$; hence by the Rule I ſet 3 on D to 7 on C, then 60 on D is off the Rule, therefore I divided it by 10, and the Quotient 6 on D is oppoſite 28 on C; then becauſe the firſt Term was divided by 10, the ſecond by *one*, and the third by *one*, the multiplying Fraction

Fraction is $\frac{1\times100}{\overline{1}^2}=100$, hence $28\times1000=2800$ is the Term ſought.

*Example* II. Let the three Terms given be 2, 80, 9, ſo that the Term ſought is a fourth Proportional to $\overline{2}^2 : \overline{80}^2 :: 9 :$ —now ſetting 2 on D to 9 on *c*, then 80 is off the line D: let it be divided by 10, the Quotient 8 is on the Line D, but does not fall againſt any Diviſion on C, whence divide 80 by 20, and the Quotient (4) on D, is againſt 36 on C, now here the multiplying Fraction is $\frac{1\times\overline{20}^2}{12}=400$, therefore 14400 is the Number ſought. But we may obſerve there is ſeldom occaſion to divide any but the ſecond Term, the Reaſon of this will be evident from hence; the three Terms given being *a*, *b*, *c* and *x*, that ſought *x* is found equal to $\frac{b^2\times c}{a^2}$. Now if *a*, *b*, *c*, are divided by *m*, *n*, *r* reſpectively, then a fourth Proportional to $\frac{a^2}{m^2} : \frac{b^2}{n^2} :: \frac{c}{r}$ is $\frac{b^2\times c\times m^2}{a^2\times n^2\times r}$, whence this multiplied by $\frac{n^2\times r}{m^2}$ gives $\frac{b^2\times c}{n^2}$, which is the fourth Proportional to $a^2 : b^2 :: c$, the Term ſought; but if the ſecond Term *b* is only divided, the Term that is then found muſt be multiplied by $n^2$ only; becauſe here *m* and *r* become each 1. We proceed in the next Place to ſhew the Uſe of the Line D E, in which the Radius of the former is triple the latter, it being laid down from a Scale three trimes as large as that of D.

The

## *The Extraction of the Cube Root by the Sliding-Rule.*

### PROP. VI.

*Any Number being given, to find its Cube-Root by the Sliding-Rule.*

The firſt 1's on the Lines D, E ſtanding againſt each other, let the Number given be found on E, then oppoſite it on D is the Root ſought. For the Radius of the line D being triple each one of E, every Number on the former is at a triple Diſtance from 1, that the ſame Number on the latter is from 1, hence the Number on D that ſtands oppoſite any Number on E, is the ſame that would be found at $\frac{1}{3}$ of the Diſtance on E, but theſe Diſtances are Logarithms of the Numbers: hence if Q be the Number given, and $q$ its Root, then $q^3 = Q$, or (*per Prop.* III. *Sect.* 2.) $3L : q = L : Q$, therefore $L : q = \frac{L : Q}{3}$, that is $\frac{1}{3}$ part of the Diſtance between 1 and Q on E, to which correſponds the Number D oppoſite Q on E is the Log. of the Root ſought, and therefore the Number it ſelf is the Root of Q. *Q. E. O.*

But if Q is ſo large that it cannot be found on E, then let it be divided by $\overline{10}^3$, or $\overline{10}^6$, or $\overline{10}^9$, and the Cube Root of this Quotient muſt be multiplied by the Cube Root of the ſaid Diviſor. One Example will be ſufficient; Let it be required to find the Cube Root of 2197: Here the firſt 1 on E being *one*, the greateſt Number on E is 1000, ſo that 2197 is not on the Line E, therefore I divide it by $\overline{10}^3$, and on D againſt the Quotient 2,197, on E is 1,3; this is multiplied by 10, the Cube

Cube Root of $\overline{10}^3$, gives 13 for the Cube Root of 2197. The Reaſon of this dividing, and its conſequent multiplying may be ſhewn after this manner; Q is the Number whoſe Cube Root is $q$, let $r$ be the Cube Root of $\frac{Q}{m^3}$, then $r^3 = \frac{Q}{m^3}$, or $rm = \sqrt[3]{Q} = q$, whence $m$ being 10, 100, 1000, &c. or $m^3$—$\overline{10}^3$, $\overline{10}^6$, $10^q$, &c. the thing is manifeſt.

## PROP. VII.

*Any three Numbers being given, 'tis required to find a fourth, which may be to the third, as the Cube of the ſecond is to the Cube of the firſt.*

*Solution.* Set the firſt on D to the third on E, then oppoſite to the ſecond on D is the fourth on E.

For let us again denote the three given Numbers by $a$, $b$, $c$, and $x$ for that ſought, then by the Condition of the Queſtion $a^3 : b^3 :: c : x$, therefore by *Sect.* II. we have $L : x = 3L : b - 3L : a + L : c$. Now when $a$ on D is ſet to $c$ on E, then $b$ on D ſtands oppoſite a Number on E, which is diſtant from 1 by the Difference of the Spaces 1, $a$: 1, $b$ on D, and the Space 1, $c$, on E; therefore theſe Spaces being Logarithms of their Numbers, the Number on E oppoſite $b$ on D is diſtant from 1 by a Space expreſſed by $3L : b - 3L : a + L : c$, to the Radius of E, but that by what was ſaid above is equal to the Logarithm of $x$, hence the Number its ſelf is equal to $\frac{b^3 \times c}{a^3}$, therefore the Proportional ſought. *Q. E. O.* If either the firſt or ſecond Terms cannot be found on D, they muſt be divided by 10, 100, 1000, &c. till they each obtain a Place thereon, and alſo till the Place of $b$ (divided) on D, is oppoſite ſome

diviſion on E, and the Anſwer that thence ariſes muſt be multiplied by a Fraction, whoſe Numerator is the Cube of the ſecond Term's Diviſor, and Denominator the Cube of the firſt Term's Diviſor; for the Terms being $a$, $b$, $c$, and the Term ſought $\frac{b^3 \times c}{a^3}$ if for $a$, $b$, $c$, we put $\frac{a}{m}$, $\frac{b}{n}$, $c$, we ſhall have the fourth Term to theſe $\frac{b^3 \times c \times m^3}{a^3 \times n^3}$, whence if this be multiplid by $\frac{n^3}{m^3}$, it will give $\frac{b^3 \times c}{a^3}$ the Term required.

*Examp.* Let the Numbers given be 2, 80, 9, here when 2 on D is ſet to 9 on E, 80 is off the Line D, but dividing it by 10, againſt 8 (the Quotient) on D is 576; whence, becauſe 80 was divided by 10, and 2 by 1, I multiply 576 by $\frac{10^3}{1^3} = 1000$, and it gives 576000 for the Term required.

To theſe three Proportionals we may add, that ſince it is demonſtrated that all ſimilar Planes are to each other as the Squares of their homologous Lines; and all ſimilar Solids as the Cubes of their homologous Lines: thence by help of theſe three Lines C, D, E, all Spaces are meaſurable which are homologous to any given Plane; and all Solids which are Homologous to a given Solid.

We are now got to the Line MD, whoſe firſt Diviſion at the right hand correſponds to ,215042, or 2,15042, or 21,5042, or 215,042, *&c.* by which means the firſt 1 thereon is againſt 2,15042, or 21,5042, or 215,042, *&c.* reſpectively on A; the Values of the Diviſions on MD increaſing towards the left hand, as thoſe on the reſt do towards the right. Why it ſhould begin thus, and not at *one*, as the others do, will be evident after we

we have laid down the two following Propoſitions, but we muſt obſerve the uſe to which this Line with the others on this Side the Sliding-Rule are applied, is in *Malt-Gauging*; the Veſſels in which that is uſually put being in the Forms of Parallelopipedons, having Rectangular Baſes. Now if the Length of any Parallelopipedon be $l$, its Breadth $b$, and Height or Depth $h$ in Inches, $l \times b \times h$ is the Number of Cubick Inches it contains, as is demonſtrated farther on; or the Area, or Meaſure of the Baſe being $q$, the Cubic Inches thereof is expreſſed by $q \times h$, but there are 2150,42 Cubic Inches in a Buſhel of Malt, therefore $\frac{b \times l \times h}{2150,42}$, or $\frac{q \times h}{2150,42}$, is the Buſhels of Malt in the ſaid Veſſel, whether Floor, Couch or Ciſtern, having the above Form.

## PROP. VIII.

*The Meaſure of the Sides of any rectangular Parallelogram being given, to find its Meaſure by the Lines* B *and* MD.

*Solution.* Set the Length on B to the Breadth on MD, then oppoſite 1 (or unity) on MD is the Meaſure on B.

For the Meaſure of the Breadth being $b$, and that of its Length $l$, the Meaſure of the Rectangle is $b \times l$, as is demonſtrated farther on, hence $q = b \times l$, or $L : q = L : b + L : l$: (*per Prop.* I. *Sect.* 2.) Now when $b$ on B is ſet to $l$ on MD, then 1 on MD is againſt a Number on B, which is diſtant from $b$ towards the right hand by the Space 1, $l$, ſo that 1 on MD is againſt a Number on B, which is diſtant from 1 by the Sum of the Spaces 1, $b$, and 1, $l$, but thoſe Spaces are Logarithms of their Numbers; whence the Diſtance of the Number

on B, oppoſite 1 on MD is expreſs'd by $L:b+L:c$; and therefore from what was ſaid above, the Number it ſelf is $L\times b$ or $q$. *Q. E. O.*

## *Of Meaſuring Floors, Couches, or Ciſterns by the Sliding-Rule.*

### Prop. IX.

*The Length, Breadth, and Height (or Depth) of any Couch, Floor, or Ciſtern being given, to find the Number of Buſhels of Malt it contains.*

*Solution.* Set the Length on B to the Breadth on MD, then againſt the Height on A is the Content on B.

For the Length being $l$, Breadth $b$, Height $h$, and the Area of the Baſe $q=l\times b$; the Content is $\frac{q\times h}{2150,42}$ as obſerved before: but by the laſt *Propoſition* $q$ is againſt 1 on MD, therefore by *Propoſition* 3, foregoing, if 2150,42 on A be ſet to $q$ on B, then oppoſite $h$ on A is $\frac{h\times q}{2150,42}$ on B. But we obſerved above, that 2150,42 on A, is oppoſite 1 on MD, and 1 on MD has been already ſhewn to be againſt $q$ on B, therefore 2150,42 on A is againſt $q$ on B, whence againſt $h$ on A is the Content on B, *viz.* $\frac{h\times q}{215042}$. *Q. E. O.*

In the Uſe of theſe Lines, A, B, and MD together, there may ariſe ſome Difficulty about eſtimating the Value of the Numbers found by theſe Directions, which muſt be cleared up before we finiſh this Chapter of the Uſe of the Sliding-Rule.

In the firſt place therefore it is to be noted, you muſt always have 1 on MD, repreſenting Unity; other-

otherways the Area of the Baſe or Product of $l \times b$ cannot be directly found on B: but you'll have $\frac{1}{100}$, $\frac{1}{10}$, ten times, 100 times, *&c.* that Area according to the Value put for the firſt 1, to the right hand MD; wherefore that the direct Value may be had, let that ever ſtand for $\frac{1}{10}$ or 1. If it repreſent 1, then the Number oppoſite on A (where is a Braſs Pin marked MB) ſhould be 2150,42, the cubic Inches in a Buſhel of Malt; but by having that ſo large, it will commonly happen that the Height or Depth which is to be found thereon, is off the Line, for the leaſt Number on that Suppoſition is 100; wherefore to remedy that, you are to diminiſh the ſeveral Meaſures by dividing them by 10, 100, 1000, *&c.* till each (when the firſt or ſecond 1 on MD is unity) may not only have a place on its reſpective Line, but that the Term required may alſo have its place on B: and whatever is the Product of all theſe Diviſors of *l*, *b*, *b*, the Number 2150,42 muſt be divided thereby, by which the Value of the Diviſors of A will be known. The Reaſon of this is thus made out, put B the Buſhels of Malt in the Ciſtern, Couch, *&c.* whoſe Length is *l*, breadth *b*, and height *b*, then $B = \frac{l \times b \times b}{2150,42}$: But let each of theſe Meaſures be divided by *m*, *n*, *r*, reſpectively, which are always 10, 100, 1000, *&c.* then I ſay, if 2150,42 is divided by the Product of theſe Diviſors, the Product of the former Quotient divided by this latter is ſtill equal to B, the Content of the ſaid Couch or Ciſtern, *&c.*

for $\frac{2150,42}{m \times n \times r}$ ) $\frac{l}{m} \times \frac{b}{n} \times \frac{b}{r}$ ( $\frac{l \times b \times b}{2150,42} = B$.

We ſhall make this Obſervation plain by Examples.

*Example* 1. Let the Breadth of a Couch be 56,2 Inches, its Length 270, and Height 5,2 Inches;

now in order that 5,2 may be found on A, the firſt 1 muſt be either $\frac{1}{10}$ or 1; let it be 1, then the ſecond MB on A is 21,5042; whence in this caſe 2150,42 is divided by 100, therefore let 270 be divided by 10, and 56,2 by 10, then ſetting 27 on B to 5,62 on MD, againſt 5,2 on A, is 36,7 on B, which is the Number of Buſhels in the Couch propoſed.

*Example* 2. Let the Breadth be 72 Inches, Length 140, and Depth 18,2. Now that 18,2 may be on A, the firſt 1 may be *one* by which the ſecond MB is 21,5042, ſo that 215042 is again divided by 100, therefore let 140 be divided by 10, and 72 by 10; then ſetting $14=\frac{140}{10}$ on B, to $7.2=\frac{72}{10}$ on MD againſt 18.2 on A, is 85,3 on B, which is the Content ſought.

*Example* 3. Let the Breadth be 13 Inches, the Length 36 Inches, and the Depth 9 Inches, therefore the firſt 1 on A may be either $\frac{1}{10}$ or 1, for in both caſes 9 may be found on A; let it be 1, then (the ſecond Braſs Pin being mark'd MB) is 21,5042, ſo that 2150,42 is divided by 100; therefore let the Breadth be divided by 10, and the Length by 10, ſo that both together are divided $10\times10=100$, and they become 1.3, and 3.6; then ſetting 3.6 on B, to 1.3 on MD againſt 9 on A, is 1.96 on B for the Content.

*Example* 4. Let the Length of a Floor be 1250 Inches, Breadth 360 Inches, and Depth 9 Inches, and let the firſt 1 on A be $\frac{1}{10}$, then the ſecond MB is 2,15042, ſo that 2150,42 is divided by 1000; therefore let 1250 be divided by 10, and 360 by 100, ſo we have 125 and 3.6; then ſetting 125 on B to 3.6 on MD againſt 9 on A, is 1884 on B, which is the Content ſought.

So that you ſee the firſt 1 to the right hand on MD has ever been *one*, the firſt *one* on B either 10, or 100, or 1000, *&c.* and the firſt 1 on A was varied

varied as occaſion required, which was always determined ſo as the Height might be found thereon, and alſo that that might fall againſt ſome Number on B; but from thence we determined by what the Diviſor 2150,42 was divided, and then the Diviſors of the Length and Breadth: For the Product of their Diviſion muſt ever be equal to the Diviſor of 2150,42. In the laſt Caſe, the ſecond Point MD being 2150,42 was taken for 2,15042; ſo that it was divided by 1000, and the Length 1250 was divided by 10, and the Breadth by 100, becauſe 10x10=1000, the Diviſor of 2150,42; ſo that from theſe Directions the Reader will never be at a loſs to eſtimate the Value of the Diviſors on the Sliding-Rule according as occaſion preſents.

The frequent uſe that is made of this Rule by my Brethren of the Exciſe, made me the fuller in this particular, that they might not only be acquainted with its Uſe; but alſo the Reaſon of the Method of Operation, which I have laid down in a manner as ſtrictly mathematical, as the Nature of the Thing admits, and yet ſo illuſtrated by verbal Rules and Examples, that I would faln hope it will not be out of the reach even of the Learner, which I think has not been done before; ſo that we ſhall diſmiſs it for the preſent till we come to the Practice, where the Reader will find a further Application of it in meaſuring the various Solids that occur in GAUGING.

# PART II.

## CHAP. I.

### *Of Geometrical Definitions and Theorems.*

IN the preceding Chapters we have laid down the Method of computing, both by the Pen and Sliding-Rule; it remains that we ſhew how thoſe Rules are applicable in finding the Contents or Meaſures of Magnitude. For as on the one hand, knowing for Example, that the Meaſure of any Parallelopipedon is equal to the Product of the Meaſures of its Baſe and Height, I ſay as this could be of little Service, without knowing how to compute by ſuch Meaſures: So on the other hand it would be of little Uſe to be able to compute without knowing when to multiply, when to divide, *&c.* and alſo the Terms by which thoſe Operations are to be made. Therefore having laid down the Method of computing, we ſhall now proceed to ſhew how thoſe Rules are to be made uſe of in Meaſuring of Magnitude.

*Definitions.*

1. *Magnitude* is whatever is extended, and therefore may be conceived to be contained or limited by ſome certain Term or Terms.

1. A *Point* is that which has no Parts, or conſidered as indiviſible, and ſo of it ſelf no Magnitude, and yet in ſome meaſure may be taken for the beginning thereof. For,

3. If we conceive A (*Fig.* 6.) to be a Point, and any how moved towards B, by this Motion we may conceive to be generated a Magnitude AB, of one Dimen-

Dimenſion only, that is, Length without Breadth, or Thickneſs; and this is called a *Line*.

4. When A moves towards B the ſhorteſt way, AB is called a *Right Line*.

5. But if the Direction of the point A be changed in every point of its Path or Way, as *Ac*, *bc*, *be*, *&c.* (*Fig.* 7.) then the generated Magnitude is a *Curved Line*; the Species of which are infinite; but ſuch of them as are uſeful in Gauging, ſhall be conſidered farther on.

6. If the Line AB be alſo conceived to move in a rectilineal Direction E *e*, its extreme Points A, B will deſcribe other Lines AC, AB, and the Line itſelf will have generated a Magnitude AD of two Dimenſions, having both *Length* and *Breadth*, but no Thickneſs; and this is called *a Plane Surface*, or ſimply a *Plane*.

7. If this Surface AD be conceived to move forward or backwards in the rectilineal Direction E *e* (*Fig.* 8.) its oppoſite Points A, D, B, C, will again deſcribe other Lines AF, DK, BH, CG, *&c.* and conſequently each of theſe Lines other Planes, and thence by this Motion we may conceive to be generated a Magnitude of three Dimenſions, *Length*, *Breadth*, and *Thickneſs*, called a *Solid*. But here as before in the Generation of Lines, ſhould the Direction of the Generating Line, or Plane, be a curve Line, then the Plane, or Solid, will accordingly be contained by a curve Line, or curved Solid.

*We gave this Generation of Magnitude, becauſe of its great Uſe and Extent in the Doctrine of curve Lines, and conſequently the whole of Mathematical Philoſophy. But yet I muſt confeſs, by thoſe of a wrangling or ſophiſtical Temper, it may be objected that we have given the Generation of a Subſtance, from a Nothing, or an Attribute, which is a ſort of Contradiction; for it's moſt certain, Points and Lines*

*are*

*are only Attributes of Body.* And ſome whoſe Geometry ſeems to be confined to ſeeing only, *have objected to the Elementor, for his negative Definitions of a Point, Line and Surface; or rather denied that ſuch Points, Lines and Surfaces exiſt at all. To the firſt it may be replied, We do not ſo much pretend to* generate Magnitude, *that would be abſurd indeed; but only to convey to the Mind the Idea of the Figure which the ſeveral Sorts of Magnitude really in being may put on. And to the ſecond we muſt obſerve, Method obliged the Elementor, whom we have imitated in this, to begin with the moſt ſimple firſt, and therefore he began with defining a Point, where he only tells us what it is not. They will reply, of nothing or non-entity there are no Properties or Affections: But we anſwer, Points or Lines are the Attributes of Body, and where that exiſts, they muſt alſo; for by firſt aſſuming ſome Body to be given, it is certain it has Ends; then if thoſe Ends are flat, or exactly even, they will be* Planes, *and thoſe Planes have alſo their Ends, which if ſtreight, they are* Right Lines, *and thoſe Lines have Ends, which are Points. So that from hence 'tis plain we have adequate Ideas of Points, Lines, and Plains, as defined by the Geometers; thoſe Points and Lines drawn with a Pen, being only Pictures of the real Points or Lines which Mathematicians conſider, and drawn to aſſiſt the Imagination.* But to proceed;

8. A *Circle* is a plain Surface, terminated by one curved Line called *Circumference*, unto which all Lines, drawn from a Point within called the *Center*, are equal to each other; theſe Lines are called *Semidiameters*, or *Radii*; and any Line paſſing thro' the Center, and terminated by the Circumference, is called a *Diameter*; the Line joining the Ends of any part of the Circumference, is called its *Chord*.

9. A *Sector* of a Circle is part thereof, terminated by two Semidiameters, and part of the Circumference, either more or leſs than the half; and this Arch is called the *Baſe* of the Sector.

10. A *Segment* of a Circle is part thereof, terminated by part of the Circumference and its Chord; and when that part is half of the Circumference, it is called a *Semicircle*.

The Ancients divided the Circumference of a Circle into 360 equal Parts, each of which they called a *Degree*; and each Degree into 60 equal Parts called Minutes; and each Minute into 60 Parts called *Seconds*, and ſo on to thirds, fourths, *&c.* every Arch of a Circle, being denoted by the Number of Degrees, Minutes, Seconds, *&c.* it contains.

11. If two Lines in the ſame Plane, and not lying in the ſame Direction, are ſo ſituate, that being produced they would meet, they are ſaid to be *inclined to each other*; and if two inclined Lines are produced till they do meet, they will form a *Plane Angle*: The Point of Concourſe is called the *Angular Point*, and the Angle is ſaid to be *more* or *leſs*, according as the Inclination of its Lines are *more* or *leſs*; but the Quantity thereof is eſtimated by the Arch of a Circle contained betwixt its Lines, and having the angular Point for its Center. Every Angle may be denoted by three Letters, the middlemoſt of which is at the Angular Point, and the other two at the Ends of the Lines. Thus (*Fig.* 9.) the Angle form'd by the Lines BD, CE produced, is denoted by BAC.

12. When a ſtraight Line ſtands on another, ſo that it does not incline to one end more than the other, but makes the Angles on both ſides equal, then are theſe called *Right Angles*; and the Lines are ſaid to be perpendicular to each other.

Hence

Hence the Meaſure of the Right Angle is 90 *Degrees.*

13. Every Angle *greater* than a Right one, is called an *Obtuſe Angle*; and every Angle leſs than a Right one, is called an *Acute Angle*; or in general both obtuſe and acute are called *Oblique Angles.* Thus,

Figure 10. { CBD, CBE, ABD } is an { Acute, Right, Obtuſe } Angle.

And the Lines AC, EB are perpendicular to each other.

14. If to one end of any Arch of a Circle a Diameter be drawn, and from the other end a Perpendicular be let fall thereon, it is called the *Sign* of the ſaid Arch. And if from that end of the ſaid Diameter, which co-incides with the Arch, a Perpendicular be erected, and a Line drawn from the Center of a Circle and the other end of the Arch, and produced till it meet the ſame; then that part thereof which is betwixt the Point of Concourſe and Center, is called the *Secant* of the Arch; and the part of the Perpendicular betwixt the Point of Concourſe and Diameter is called the *Tangent* of the Arch. Alſo that part of the Diameter betwixt the Sine and Tangent of any Arch, is called its *Verſed Sine*: Thus in the Circle ACBA, (*Fig.* 11.)

{ CEB, CEA } are Sectors.
{ BHED, AKEA } are Segments.

ABIA is a Semicircle.

{ ES, BT, CT, BS, AS } is the { Sine, Tangent, Secant, Verſed Sine } of the Arch EB.

15.

15. Thoſe Right Lines drawn on the ſame Plane are ſaid to be parallel, which being produced ever ſo far would not meet, or the Parallels are ſuch, that if from ever ſo many Points in the one, Perpendiculars be let fall on the other, they will be all equal to one another. Thus, if AB and CD, (*Fig.* 12.) are two Lines ſo ſituated to one another, that from the Points *a*, *a*, *a*, in AB, the Lines *ab*, *ba*, *ab*, being let fall perpendicular on C, the ſeveral Lines *ab*, *ab*, *ab*, are equal among themſelves, then is AB parallel to CD.

*Of Right-Lined Surfaces, or Planes.*

16. *Right-Lined Surfaces* are thoſe Planes which are included by Right Lines, and the including Lines are called their Sides.

17. A *Triangle* therefore is a Plane included by three Right Lines, of which reſpecting its Sides, that is *equilateral*, whoſe three Sides are equal, (*Fig. a.*) that an *iſoſceles Triangle* having two Sides equal, (*Fig. b.*) and a *ſcalene*, or *oblique Triangle*. when all the Sides are unequal. (*Fig.* 16.) But reſpecting its Angles, that is a *right-angled Triangle* which has a right Angle, (*Fig. c.*) and an acute-angled Triangle when all the Angles are acute, (*Fig. a.*) and an obtuſe *angled Triangle*, when it has one obtuſe Angle. (*Fig.* 16.)

18. In right-angled Triangles, the two Sides containing the right Angle are called its Legs, and the Side ſubtending the right Angle is called the *Hypothenuſe*.

19. If a Perpendicular be let fall from any Angle of a Triangle to its oppoſite Side, that Side is then called the *Baſe* of the Triangle; the Perpendicular its *Height*, and the Point from whence the Perpendicular was drawn, is called its *Vertex*.

20. Of *four-ſided Figures*, thoſe are called *Parallelograms*, whoſe oppoſite Sides are parallel or equal;

equal; and theſe are diſtinguiſhed into four Sorts; 1. When the Sides are all equal, and Angles right, they are called *Squares*, (*Fig.* 13.) 2. When the oppoſite Sides are only equal, and Angles Right, *Rectangles*, (*Fig.* 14.) 3. When the Sides are equal, and Angles oblique, *Rhombus's*, (*Fig.* 17.) 4. When the oppoſite Sides are only equal, and Angles oblique *Rhomboids*, (*Fig.* 15.) All other quadrilateral Figures called *Trapezias*, (*Fig.* 18.)

*Of multilateral or polygonal Figures.*

21. A *Pentagon* is contained by five equal Sides, (*Fig.* 20.) A *Hexagon* by ſix. A *Heptagon* by ſeven. An *Octagon* by eight, *&c.* Each taking its Name from the Number of equal Sides or Angles by which it is contained.

To this Definition we may add, that all Polygons may be circumſcribed by a Circle, or have a Circle inſcribed in them: For if any two contiguous Sides of a Polygon are perpendicularly biſſected, the cutting Lines will meet in a Point, which is the Center both of the circumſcribed and inſcribed Circles; the Diſtance of that Section, and any Angle of the Polygon, being the Radius of the former, as the Diſtance of the ſaid Section, and middle of any ſide is of the latter: Whence the Center of the inſcribed or circumſcribing Circle of any Polygon, is called the Center thereof; and that not improperly, it being its Center of Gravity. Thus (*Fig.* 20.) DC and EC being drawn perpendicularly thro' DE, the middle of AB, BE, interſect each other in C, the Center of the Polygon; whence CB is the Radius of the circumſcribed, and CD=CE of the inſcribed Circle.

22. All multilateral Figures of unequal Sides, are called *irregular Polygons*.

23. The Length of a Perpendicular drawn from any Angle of a Parallelogram to its oppoſite Side,

is

is called its *Height*; and the Side on which it falls is then called its *Baſe*.

We gave theſe elementary Definitions for the ſake of thoſe unskill'd in geometrical Subjects; but when we have occaſion, in what follows, to mention any Properties of the things defin'd above, we muſt refer the Reader to *Euclid*; for our intended Brevity would not permit giving a regular Method of Demonſtration, and therefore we chuſe to omit them altogether.

*Of the Meaſure of Magnitude.*

23. Every Magnitude is meaſured by another of the ſame kind: As a Line is meaſured by a Line, a Surface (or Plane) by a Plane, and a Solid by a Solid: And the ſame extends to Velocity and Tardity in Motion, Intention, and Remiſſion in Qualities, and ſo on, of every thing to which the Words *more* or *leſs* can be apply'd.

24. And that Magnitude by which another is meaſured, we call its *meaſuring Unit*. This is arbitrary, and may be aſſigned by the contracting Parties at pleaſure. *The meaſuring Unit for Lines may be a Line of any Length*; but for ſhort Lines is commonly a Foot, or twelve Inches. For *Surfaces*, the meaſuring Unit is a Square, whoſe Side is the *meaſuring Units for Lines*. And the meaſuring Unit for Solids is a Cube, *whoſe Side is equal to the meaſuring Unit for Surfaces*.

25. The Meaſure of any Magnitude is the *Ratio* it has to the meaſuring Unit, or may be ſaid to be the Number of Times and Parts of a Time that it contains the meaſuring Unit. Thus if the Line *ab* (*Fig.* 13.) be taken for the meaſuring Unit, and the Line AB be four times as long, then its Meaſure is 4; for AB : *ab* : : 4 : 1, or $\frac{AB}{ab} = \frac{4}{1} = 4$.

Again, if *a b c d* (*Fig.* 14.) be a Square, whoſe Side is equal to *ab* in the laſt, and A B C D is a Space

Space 12 times as much; then the Meaſure of ABCD is 12; for ABCD : *abcd* : : 12 : 1, or $\frac{ABCD}{abcd} = \frac{12}{1} = 12$. *So that from hence we may ſee by the Meaſure of any Magnitude, is meant its Numeral Relation to another of the ſame kind eſteemed as Unity.*

## *Of meaſuring right-lined Surfaces.*

### Prop. I.

*The Meaſure of every Rectangle, is equal to the Product of the Meaſures of its containing Sides.*

Let the Meaſure of BD, one of the Sides of the Rectangle ABCD (*Fig.* 14.) be called $l$; and the Meaſure of AB, (the other containing Side) $b$; then if B*a*, *ag*, *gh*, &c. are taken each equal to the linear Unit, their Number will be $l$: through each of which draw Lines parallel to BD, meeting DC in as many Points *r*, *s*, *t*, &c. then if B*d* and *a c* are each taken equal to B*a*, 'tis evident (from *Definit.* 24.) B*acd* will be the ſuperficial Unit; and let *d e*, *e f*, &c. be taken each equal to B*d*: Now 'tis manifeſt the Meaſure of B*ar*D will be $l \times 1$, for B*ar*D : B*acd* : : $l \times 1$ : 1; whence (*per Definit.* 25.) $l \times 1$ is the Meaſure of B*ar*D; but the Spaces B*r*, *as*, *g t*, &c. are equal (by 36. 1. *Eucl.*) therefore any one of them ſo often repeated as is their Number, will be the Meaſure of the whole Figure, but $b$ is put for that Number, therefore $b \times l \times 1 = bl$, *viz.* the Product of the Meaſures of the contained Sides, is the Meaſuree of the Rectangle ABCD. *Q.E.D.* But if betwixt the Points B*a*, any Point *v* was taken, and the Line *v w* drawn parallel to BD, ſo that a rectangular Space BD*wu* was to be meaſured, whereof one ſide was leſs than

than the linear Unit, the Meaſure of that Space will be expreſſed as a Fraction, whereof the Meaſure of the Side BD is Numerator, and that of $\frac{1}{Bv}$ Denominator; for let $\frac{1}{p}$ expreſs what part B*v* is of B*a*, or Unity, then the Meaſure of B*v* is $\frac{1}{p}$, and (from the 1. 6th *Euclid*) BD*wv* is the ſame Part of BD*ra* that B*v* is of B*a*, but that is $\frac{1}{p}$ by ſuppoſition; and from what was ſaid above, the Meaſure of BD*ra* is *l*×1, or *l*; therefore B*vw*D=$l \times \frac{1}{p}$, *viz.* equal to the Product of the Meaſures of the containing Sides, whence the Propoſition is true univerſally.

*Scholium.* Multiplication, properly ſpeaking, can only obtain where theMultiplier is a whole abſtract Number; but for want of a better Word, it is commonly uſed tho' that was a Fraction, or Surd, becauſe thence is produced a Number, which has the ſame Ratio to one Factor, that the other has to Unity; which is alſo the caſe when they are whole Numbers; for the Factors being *f*, F, and the Product *p*, we have $f \times F = p$, or $p : f :: F : 1$.

Cor. 1. *Hence the Meaſure of every Parallelogram is equal to the Product of the Meaſure of one of its Sides, into a Perpendicular let fall thereon from the other.*

Thus (*Fig.* 15.) put *p* for the Meaſure of the Perpendicular AP, and *s* for the Meaſure of the Side CD=AB, then *p*×*s* is the Meaſure of ABCD; for if CD be continued to Q, where it meets a Line drawn thro' B parallel to AP, then (by the 35 *Euclid.* 1.) APQB is equal to the Parallelogram ACDB; but the Meaſure of the former is $p \times s$, and therefore $p \times s$ is alſo that of the latter: *Whence to meaſure any parallelogramick Space by the*

*Sliding-Rule, we have this Rule:* Set 1 on B to the Length on A, then oppoſite to the Breadth on B, is the Content on A, (by *Prop.* 1. *Sect.* 2.)

*Example.* Let AB=12.55 Inches, and AC=8.32: then the Meaſure of the ABCD is 12.55×8.32= 104,416 ſquare Inches. Or, *by the Sliding Rule*, 1 on B ſet to 12.55 on A againſt 8.32 on B, is 104.416 on A, which is the Meaſure ſought.

Cor. 2. *Hence the Meaſure of every right-lined Triangle is equal to half the Product of the Meaſure of one of its Sides into the Meaſure of a Perpendicular drawn thereon, from its oppoſite Angle.*

Let the Meaſures of the Sides CD, DB, BC, (*Fig.* 16.) of the Triangle BCD be $a$, $b$, $c$, reſpectively, and the Meaſures of the Perpendiculars BQ, CP, DR, $p$, $q$, $r$, reſpectively, then the Meaſure of BCD is $\frac{a \times p}{2}$, or $\frac{b \times q}{2}$, or $\frac{c \times r}{2}$; for thro' B draw a Line parallel to CD, and another thro' C, parallel to BD, meeting the former in A, then the Parallelogram ABCD is double the Triangle BCD (*per* 42 *Euclid.* 1.) but the Meaſure of ABCD by the preceding *Corol.* is $a \times p$, therefore $\frac{a \times p}{2}$ is the Meaſure of the latter. The other two Expreſſions may be proved after the ſame manner.

*Cor.* 3. From hence we deduce the manner of meaſuring all right-lined Spaces univerſally; which is done by dividing the Figure into the leaſt Number of Triangles poſſible, theſe will ever be *leſs by two* than the Number of the Figure's ſides; and the Meaſure of each of them being found by the laſt *Corol.* their Sum will be the Meaſure of the Figure propoſed. But that may be effected at one Operation, by firſt reducing the given Figure into a Triangle; the Method for doing which will be manifeſt from this,

PROBLEM.

### PROBLEM.

*To reduce any Trapezia, or Figure, contained by four right Lines into a Triangle.*

Let the four-ſided Figure be ABCD, join any two oppoſite Angles as B, D, by the Line BD; alſo thro' C, or A, the other Angles, draw a Line parallel to the Diagonal Line BD; let it meet the Sides AD or CB produced in E or F, and draw the Line BE or DF, the firſt of which cuts CD in K, and the ſecond AB in L, then ſhall the Triangles BAE and CDF be each equal to ABCD: for the Triangle CDE is equal to CBE, (*per* 36 *Eucl.* 1.) therefore the Triangle DKE is equal to BCK, to which BADK being added, we have BAE equal to ABCD; and after the ſame manner CDF is proved equal to ABCD. *Q. E. F.*

Therefore if the Figure has five Sides, firſt divide it into a four-ſided Figure and a Triangle, by drawing a Line from any Angle to the next but one: the former of which is reduced to a Triangle by the above Problem, and this Triangle, with what the ſaid four-ſided Figure wants of that given, will form another four-ſided Figure; which being reduced by the above Problem, that will be the Triangle ſought: And after the ſame manner you may proceed if it has ſix, ſeven, eight, or ever ſo many ſides, which the Reader will more readily perceive by conſulting the 18 and 19 *Figures*; the firſt ABCDE is a five-ſided Figure, and the ſecond ABCDEF has ſix, which are reduced into the Triangles AFG, GHK.

Cor. 4. *From* Cor. 2. *we alſo infer that the Meaſure of every Polygon, is equal to the Sum of the Meaſure of its Sides into the Meaſure of a Perpendicular thereon from the Center.*

But ſince the Perpendicular from the Center of a Polygon is required to find its Area, we ſhall

lay down a Method by which the Area of any Polygon may be directly found from its Side being given: In order to which, let $n$ denote the Number of Sides of the Polygon, and from the Table of natural Signs take the Sine and Coſine of $\frac{180}{n}$ Degrees; then as four times the Sign of $\frac{180}{n}$ Degrees, is to the Coſine thereof, multiply'd by $n$, ſo is the Square of any Polygon's Side, to its Area: For the Product of the Sine and Coſine of $\frac{360}{2n}$ or $\frac{180}{n}$ Degrees, is the Area of a Triangle in a Polygon, the Radius of whoſe circumſcribed Circle is 1, ſimilar to any other Polygon of $n$ Sides; but like Triangles are as the Squares of their homologous Sides, and twice the Sine of $\frac{180}{n}$ Degrees is the Baſe of the Polygon, the Radius of whoſe circumſcribed Circle is Unity. Hence if the Sine and Co-ſine of $\frac{180}{n}$ are called $s$, $c$, reſpectively, alſo S the Side, and A the Area of the Polygon given, then $4s^2 : s \times c \times n :: S^2 : A$. or $4s : cxn :: S^2 : A$.

From whence we deduce this Rule; *To find the Area of any Polygon having its Side given.* Let 180 Degrees be divided by the Number of the Polygon's Sides; and let $n$ be divided by four times the Tangent of that Quotient (taken from the Table of natural Tangents) then if the Quotient of that Diviſion is multiply'd by the Square of the Side of the Polygon given, the Product will be the Meaſure of its Area.

*Example.* Let the Side AB (*Fig.* 20.) of a Pentagon be 50 Inches, and its Area or Meaſure required; here, according to the Rule $\frac{180}{5} = 36$, whoſe

whoſe Tangent from the natural Tables is ,7265435 by four Times, which (*viz.* 2,906178) $n=5$ being divided, we have 1.720477 for the Quotient; and this multiplied by 2500, the Square of the Side (50) it gives 4301,175 ſquare Inches for the Area. But becauſe it would be troubleſome always to have Recourſe to the Table of natural Tangents, I have thence ſelected the Tangents of $\frac{180}{3}=60$, $\frac{180}{4}=45$, $\frac{180}{5}=36$, $\frac{180}{6}=30$, *&c.* Degrees, which anſwer to half the Angles at the Center of a Polygon of 3, 4, 5, 6, 7, *&c.* Sides; as alſo the Quotients ariſing from dividing $n$ by the Quadruples thereof, as in the following Table.

*A Table for meaſuring* POLYGONS.

| The Names of the Polygons. | N° of Sides. | Half the Angle at the Center. | The Natural Tangent of the laſt Column. | Second Column divided by Quadruple the 4th. |
|---|---|---|---|---|
| *Equilat. Triangle* | 3 | 60° 00″ | 1.7325058 | .433013 |
| *Square* | 4 | 45 00 | 1.0000000 | 1.000000 |
| *Pentagon* | 5 | 36 00 | .7265425 | 1.720477 |
| *Hexagon* | 6 | 30 00 | .5773502 | 2.598076 |
| *Heptagon* | 7 | 25 42 6/7 | .4815712 | 3.633959 |
| *Octagon* | 8 | 22 00 | .4040262 | 4.828427 |
| *Nonagon* | 9 | 20 00 | .3639702 | 6.181824 |
| *Decagon* | 10 | 18 00 | .3249196 | 7.694209 |

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This Table may be continued at pleaſure, by the Rule laid down for that purpoſe, but from this only the Area of any regular Polygon of leſs than eleven Sides is had directly by a ſimple Multiplication, as we explained before. But from hence we alſo may give the Method of Meaſuring any of thoſe Polygons by the Lines C, D, on the Sliding-Rule; for by calling any Term in the fifth Column *q*, and the Side of the Polygon S, by this Proportion the Area is found, *viz.* $1 \times 1 : q :: S^2 :$ (α) the Area: therefore (by *Prop.* 5. *Sect.* 2. *Part* 1.) we have this Rule: *Set* 1 *on the Line* D *to* (q) *any Number in the fifth Column on* C, *then on* C *oppoſite the Side of the Polygon given on* D, *is the Content or Area ſought.*

Thus in the laſt Example, the Side of a Pentagon was 50 Inches, therefore becauſe the Number in the fifth Column oppoſite Pentagon is 1.720477, I ſet 1 on D to 1.72 on C, then oppoſite 5 on D (for 50 is of the Rule) is 43.01 on C, and becauſe 50 was divided by 10, I multiply 43,01 by 100, (the Square of 10) and find 4301 for Area ſought. See *Prop.* 5. *Sect.* 2. *Part* 1.

Having now ſhewn how to find the Meaſures of Plains contain'd by right Lines only, we ſhould proceed to thoſe contained either wholly by a curve Line, or partly by a curve Line, and partly by a ſtraight one; but firſt we muſt lay down the following Properties of the Conic Sections, becauſe every Reader whoſe Buſineſs requires Skill in Gauging, cannot be ſuppoſed furniſh'd with a Treatiſe of that kind, they not being ſo commonly read as the *Elements of Euclid.*

## CHAP. II.

### *Of the Elements and ſome of the eſſential Properties of the Conic Sections.*

*Defin.* 1. IF (AC Fig. 21, 22, 23, 24, 25,) one Leg of a right-angled Triangle ABC remains fixt, and the whole Triangle be conceived to turn thereon as an Axis, till it return to the Place from whence it moved, it will generate the likeneſs of a Solid called a *right Cone*; of which the fixt Side AC is called the *Axis*, the other Leg BC deſcribes a Circle called the *Cone's Baſe*; and AB the Hypothenuſe will deſcribe the *Conic Surface*. A, the Vertex of the generating Triangle, being alſo the Vertex of the generated Cone.

2. If the Hypothenuſe AB ſhould paſs the Vertex A, the external Part AB will generate another ſimilar Cone A*bd*, and both together are called *oppoſite Cones*. See *Fig.* 25.

*Cor.* 1. Hence every Plane drawn thro' the Vertex and Center of the Baſe is perpendicular thereto, (18 *Euclid* 11.) for it paſſes thro' the Axis AC, which by Suppoſition is perpendicular to the Baſe of the Cone,

Cor. 2. *Hence the common Section of every Plane drawn thro' the vertex, with the Cone, is a right-lined Triangle.* For let ABMDN (*Fig.* 1.) be the Cone, and AMN the Section made by a Plane paſſing the Vertex A; alſo let C be the Center of the Cone's Baſe, draw the Lines CM, and CN, then 'tis evident the two right-angled Triangles ACM and ACN are the Poſitions of the generating Triangle, when the Point B is in M, N, reſpectively; therefore their Sections with the Plane AMN, are two Right AM, AN, (8 *Euclid.* 11.) which are alſo

the Sections of the ſaid Plane and Cone; and MN the Section of the ſaid Plane and Baſe, is alſo a right Line, wherefore AMN the Section of the Cone by a Plane thro' its Vertex, is a right-lined Triangle.

Cor. 3. *Hence alſo the Section of the Cone, by a Plane Parallel to its Baſe, will be a Circle.* For (*Fig.*22.) let *o* be any Point in the Axis of the Cone, thro' which draw a Line *o* F, parallel to the Baſe of the Triangle, CB, then 'tis manifeſt while the Side BC deſcribes the Circle BMDN, which is the Cone's Baſe, *o* F will deſcribe another Plane ſimilar to it, but *o* F in every Point of its Path retains its Paralleliſm to BC; wherefore in its Motion it will deſcribe the ſame Figure that would be made by a Plane paſſing thro' *o*, parallel to the Baſe of the Cone; but the former has been ſhewn to be a Circle, therefore ſo is the latter.

*Defin.* 3. If a Cone be any how cut by two Planes, one thro' its Vertex and Center of the Baſe making the Triangle ABD, (*Fig.* 23.) and another perpendicular thereto, and ſo as it cuts both the Sides of the Triangle ABD, but oblique to the Baſe BD, the Figure of the Section is a curve Line called an *Ellipſe*; if V*v*, denotes the Line where the elliptic Plane cuts the Triangle ABD, then V *v*, is called the *tranſverſe Axis* of the Ellipſe; and V, *v*, its *Vertexes*.

*Defin.* 4. The Parts of Lines drawn within the Ellipſe at right Angles to the Tranſverſe, and intercepted by the ſame and Curve, are called *Ordinates* to the Axis. The Middle of the Tranſverſe is the Center of the Ellipſe, and that double Ordinate which paſſes thro' the Center, is called the *Conjugate Axis*.

*Defin.* 5. The Diſtance betwixt the Vertex of an Ellipſe,

Ellipſe, and the Section of the Tranſverſe, and any Ordinate, is called an *Abſciſſa* thereof.

*Defin.* 6. If a Cone be cut by two Planes, one any how thro' its Vertex and Center of the Baſe, making the Triangle ABD, (*Fig.* 24.) and another at right Angles to this, and parallel to one Side of the ſaid Triangle, the Figure of the Section is a curved Line called a *Parabola*, the common Section of the parabolic and vertical Plane ABD, *viz.* VO, is the Axis thereof, and V the Section of a Side of the Triangle ABD and parabolic Plane, the *Vertex* of the Section.

*Defin.* 7. The Parts of Lines drawn in the Parabola at right Angles to the Axis, intercepted betwixt the ſame and Curve, are called *Ordinates* of the Axis.

*Defin.* 8. The Diſtance on the Axis, betwixt the Vertex and any Ordinate, is the *Abſciſſa* thereof.

*Defin.* 9. If oppoſite Cones be cut by two Planes, one any how thro' the Vertex and Center of the Baſe, making the Triangles ABD, A*bd*, (*Fig.* 25.) and the other at right Angles to this, ſo as to cut the oppoſite Cones (on both Sides the Vertex A) the Figure of the Sections made in the Conic Surfaces, are curved Lines called *Oppoſite Hyperbola's*; and each of them an *Hyperbola*. Let V, *v*, be the Sections of the Cutting-Plane with AB, AD the Sides of the Triangle ABD produced, then V *v*, is called the *Tranſverſe Axis* of each Hyperbola, V, *v* its *Vertexes*; and VO the Section of the Plane ABD, and Hyperbola, is called its *Axis*.

*Defin.* 10. The Parts of Lines drawn within the Hyperbola, at right Angles to the Axis, intercepted by the ſame and Curve, are called *Ordinates* of the Axis.

*Defin.* 11. The Diſtance on the Axis betwixt the Vertex and Section of the Axis and any Ordinate,

dinate, is called the *Abſciſſa* thereof. Hence in the Ellipſe and Hyperbola, every Ordinate has *two* Abſciſſa's, and in the Parabola but *one*.

*Defin.* 12. The *Ellipſe*, *Parabola*, and *Hyperbola*, are called the *Conic Sections*. For tho' the Triangle and Circle may alſo be had by cutting a Cone, (as has been already obſerved) yet each of theſe having other Deſcriptions, and Names previouſly aſſigned them, the Ancients gave the Name of *Conic Section*, only to the three Figures mentioned in this Definition.

*Defin.* 13. In the Figures 23, 24, 25, let ABD be the Section of the Cone, by a Plane thro' the Vertex and Center of its Baſe, V*v* the Axis of the Sections, which (produced if needful) meets the common Section of the Baſe and vertical Plane ABD, (*viz.* BD) in O; and let a Line be drawn thro' V, parallel to BD, meet AD in G; then a fourth Proportional to the Lines VO, OB, and VG, is called the *Parameter*, or *Latus Rectum* of the Section, whether Ellipſe, Parabola, or Hyperbola. Whence if *p* be put for the Parameter, then VO : OB : : VG : $p = \frac{VG \times OB}{VO}$.

*Defin.* 14. The Plane ABD, which in every Section is drawn thro' the Vertex and Center of the Cone's Baſe, and perpendicular to the Conic Section, we will for Brevity ſake call the *Vertical Plane* of that Section.

## PROP. I.

*In every Ellipſe it will be as the Tranſverſe Axis is to the Parameter, ſo is the Rectangle of the Abſciſſa of any Ordinate to the Square of that Ordinate.*

Let ABD (*Fig.* 23.) be the Section of the Cone thro' its Vertex, and the Center of the Baſe, and

V*mvn*

V*mvn* the Ellipſe given, whoſe Axis is V*v*; take any Point *o* in the Axis, thro' which let a Plane paſs parallel to the Baſe BMDN, meeting the Ellipſe in the right Line *m o n*, and the Triangle ABD in EF; draw GV parallel to BD, and continue V*v* till it meet BD (the Section of the vertical and elliptical Planes) in O.

Then becauſe both V*m v n* and E*m* F*n* are perpendicular to ABD, their common Section (*m n*) will alſo be perpendicular to ABD. (18 *Euclid.* 11.) Therefore *mn* is perpendicular both to EF and V*v*, whence *m o* and *n o* are Ordinates of the Abſciſſæ V*o*, *vo*, (*Defin.* 4. *Euclid* 11.) but E*m* F*n* is a Circle, (by *Cor.* 2. *Defin.* 2.) therefore from the (8. 6. *Euclid*) $\overline{om}^2 = Eo \times oF$, but the Triangles V *o*F, VOB are ſimilar by Conſtruction; therefore V*o* : *o*F :: V O : O B, (4 *Euclid* 6.) hence $oF = \frac{Vo \times OB}{VO}$ (16 *Euclid* 6.) Again, the Triangles *vo*E, *v*VG, are ſimilar, therefore *vo* : *o*E ; ; *v*V : VG, whence $oE = \frac{vo \times VG}{vV}$, therefore $oE \times oF = \frac{Vo \times vo \times OB \times VG}{VO \times vV}$ but $Eo \times oF = \overline{om}^2$, conſequently $\overline{om}^2 = \frac{Vo \times vo \times OB \times VG}{VO \times vV}$ but $\frac{VG \times OB}{VO} = p$, (by *Defin.* 13.) therefore $\overline{om}^2 = \overline{on}^2 = \frac{Vo \times vo \times p}{Vv}$, or V*v* : *p* : : V*o*x*vo* : $\overline{mo}^2 = \overline{on}^2$. Q. E. O.

*Cor.* 1. Therefore if the tranſverſe Axis of an Ellipſe is called *t*, any Ordinate *o m*, or *o n*, *y*, and either of the Abſciſſæ thereof, *viz.* *v o*, or VO, *x*, then by this Propoſition we have $t : p :: \overline{t-x} \times x : y^2 = \frac{p}{t} \times \overline{tx-x^2}$, which may be called the Equation of the Ellipſe.

*Cor.*

*Cor.* 2. In this by making $x=\frac{t}{2}$, and calling the conjugate Axis $c$, then $y$ becomes equal to $\frac{c}{2}$, whence we have $\frac{c^2}{4}=\frac{p}{t}\times\overline{\frac{tt}{2}-\frac{t^2}{4}}$, or $p=\frac{c^2}{t}$, put this for $p$ in the above expreſſion, and we find $y^2=\frac{c^2}{t^2}\times\overline{tx-x^2}$, which is a *ſecond Equation of the Curve.*

*Cor.* 3. Hence the Parameter of the Axe of every Ellipſe is a third Proportional to the tranſverſe and conjugate Axes.

*Cor.* 4. If a Circle be deſcribed on the tranſverſe Axis of any Ellipſe, and an Ordinate be drawn to it, alſo cutting the Ellipſe, then it will be as the tranſverſe Axis is to the Conjugate, ſo is the circular Ordinate to the elliptic Ordinate; that is (*Fig.* 33.) $Vv : MM :: NO : nO$, for from the Property of the Circle $VO\times Ov=\overline{ON}^2$; and by this *Propoſition* $VO\times Ov : \overline{nO}^2 :: \overline{Vv}^2 : \overline{MM}^2$, hence $\overline{NO}^2 : \overline{nO}^2 :: VO\times vO\times\overline{Vv}^2 : VO\times vO\times\overline{MM}^2$, that is $NO : nO :: Vv : MM$.

## Prop. II.

*In every Conic Parabola it will be as the Parameter is to any Ordinate, ſo is that Ordinate to its Abſciſſa.*

Let ABD (*Fig.* 24.) be the vertical Plane, or Section of the Cone, through its Vertex, and the Center of its Baſe, MVN the parabolic Plane whoſe Axis is VO, the Vertex being V; in the Axis, aſſume any Point $o$, through which let a Plane be drawn parallel to the Baſe, whoſe Section with the parabolic Plane is the right Line $mon$, with the Triangle ABC, the right Line EF, and with the Conic Surface, the Circle EmFn,

E*m*F*n*, whoſe Diameter is EF, draw VG parallel to BD, then 'tis manifeſt from what was ſaid in the laſt Propoſition; *mn* is perpendicular both to EF and VO, therefore E*m*F*n* being a Circle E*o*×*o*F=$\overline{om}^2$, but the Triangles V*o*F, VOB are equiangular, whence VO : OB :: V*o* : *o*F, therefore *o*F=$\frac{OB \times Vo}{VO}$, but E*o*=GV, becauſe they, as alſo EG and V*o* are parallel, whence *o*F×*o*E=$\frac{OB \times Vo \times GV}{VO}$, but $p=\frac{OB \times GV}{VO}$ (*per Defin.* 13.) whence *o*F×*o*E=*p*×V*o*, but *o*F×*o*E=$\overline{mo}^2=\overline{on}^2$, therefore $\overline{om}^2=p \times VO$, that is *p* : *om* :: *om* : V*o*. *Q. E. D.*

*Cor.* 1. Therefore *p*×VO=$\overline{OM}^2$, hence VO : V*o* :: $\overline{MO}^2$ : $\overline{mo}^2$, that is in Parabola, *the Abſciſſa's are as the Squares of their Ordinates.*

*Cor.* 2. Let V*o* be callled *x*, and *om*, *y*, then from this *Propoſit.* $px=y^2$, which may be called the *Equation of the Parabola.*

*Cor.* 3. Hence, if thro' any Point of the parabolic Curve, a Line be drawn parallel to the Axis thereof, thereby cutting the Baſe unequally, *the Rectangle of the Parts of the biſected Baſe, will be equal to the Rectangle of the ſaid parallel Line, and the Parameter of the Parabola.*

For (*Fig.*44.) draw the Ordinate *vw*, then by this *Propoſition* *p*×VO=$\overline{OM}^2$, and *p*×V*w*=$\overline{vw}^2$, hence *p*×VO−V*w*=$\overline{MO}^2-\overline{vw}^2$, that is *p*×*vo* = $\overline{MO+vw} \times \overline{MO-vw}$=N*o*×*o*M.

PROP.

## PROP. III.

*In every Conic Hyperbola it will be as the tranſverſe Axis is to its Parameter, ſo is the Rectangle of the Abſciſſa's of any Ordinate to the Square thereof.*

Let ABD and A*bd* (*Fig.* 25.) be the Section of the oppoſite Cones by a Plane thro' their common Vertex A, and the Axis of the Cone meeting the Baſes thereof in the right Lines BD, *bd*, and let MVN denote the Section of the Cone, by a plain Perpendicular to ABD, which being produced, would alſo cut the Triangle A*bd* in *v*, then MVN *mv n* are Hyperbola's, whoſe Vertexes are V, *v*, and the tranſverſe Axis V*v*: in VO the Hyperbolic Axis aſſume any Point *o*, thro' which let a Plane be drawn parallel to the Baſe of the Cone, meeting the hyperbolic Curve VMN in the right Line *m o n*, the Triangle A B D in the right Line EF, and the Conic Surface in the Circle E*m*F*n*: alſo draw VG parallel to EF or BD: then again *o m* or *on* will be equal and perpendicular both to the Axis VO, and the Diameter EF, whence *o m* and *o n* are each Ordinates of the Abſciſſæ VO, *vo*; (whence by the 8. 6. *Eucl.*) E*o* × *o*F = $\overline{mo}^2$ = $\overline{on}^2$; but the Triangles VOB, V*o*F, are ſimilar, and ſo are *v o* E and *v*VG, therefore VO : OB :: V*o* : *o* F; therefore *o* F = $\frac{V o \times OB}{VO}$; alſo *v*V : VG :: *v o* : *o*E, hence *o* E = $\frac{v o \times V G}{v V}$, conſequently *o*E × *o*F = $\overline{mo}^2$ = $\overline{on}^2$ = $\frac{V o \times o v \times OB \times VG}{v V \times VO}$, but $\frac{OB \times VG}{VO}$ = *p*, therefore $\overline{mo}^2$ = $\overline{on}^2$ = $\frac{V o \times o v \times p}{V v}$, that is V*v* : *p* :: V*o* × *ov* : $\overline{mo}^2$ = $\overline{on}^2$. *Q. O. E.*

*Cor.*

*Cor.* 1. Therefore call the tranſverſe Axis of the Hyperbola, *viz.* $Vv=t$, the Abſciſſa $Vo$, $x$, and the Ordinate $mo=on$ $(=)$ $y$, then $vo=t+x$, whence $y^2=\frac{p}{t}\times t+x\times x$, which may be called the *Equation of the Hyperbola.*

From theſe Properties of the Conic Sections we might deduce a great Number of others, for they are the Foundation of all the reſt; alſo by the help of a few Lemma's the Elliptic or Hyperbolic Spaces might be compared together, as is done by that great Promoter of Mathematicks in *France*, the late Marq. *de l'Hoſpital*, tranſlated into *Engliſh* by my ingenious Friend Mr. *Edm. Stone*, known to the mathematical World by the Tranſlations he has made of ſeveral curious Treatiſes from the *Latin* and *French*, to which I ſhall refer for any other Properties of the Conics I may want in the following Work, which the Limits I have preſcribed my ſelf, would not allow room for here. But before I finiſh this Chapter, it will not be amiſs to give the Method of deſcribing theſe Curves *in Plano*, ſince ſome may be deſirous not only to meaſure them and the Solids thence form'd, but alſo to give a Deſcription thereof.

## *Of deſcribing the Conic Sections* in Plano.

### Problem I.

*The Parameter of any Parabola being given, to deſcribe the ſame* in Plano.

Draw two indefinite Lines AB, CD, (*Fig.* 26.) perpendicular to each other, in any Point V, aſſumed for the Vertex of the Section; then on the ſecond (CD) from V, ſet off VP equal to the Parameter given, and from P aſſume any Number of

of Points *c*, *c*, *c*, *c*, *c*, on which with the Diſtances *c*P, *c*P, *c*P, *&c.* deſcribe the Circles P*ss*, P*ss*, P*ss*, *&c.* cutting the Line AB in the Points *s*, *s*, *s*, *s*, *s*, *s*, *&c.* and the Line CD in the Points P, and *o*, *o*, *o*, *o*, *&c.* thro' which draw Lines parallel to AB; then if thro' each of the Points *s*, *s*, *s*, *s*, *&c.* Lines are drawn parallel to CD, they will cut thoſe parallel to AB, in *m*, *m*, *m*, *m*, *&c.* Points of the Curve required; which connected with a regular Curve-Line, it will be the Parabola ſought.

For 'tis manifeſt (*8 Eucl.* 6.) the ſeveral Lines V*s* are mean Proportionals betwixt PV, and the correſponding Line V*o*; therefore if the Parameter VP is called *p*, any of the Lines V*o*, *x*, and its correſponding *om*, *y*, we have $px=y^2$, which is the Equation of the Curve deſcribed, whence *m*V*m* is a Parabola, *Cor.* 1. *Propoſit.* 2. Q. E. F.

By this Conſtruction a Parabola may be deſcribed with great facility by help of a parallel Ruler, and is alſo applicable, when the Parameter of any other Diameter as well as that of the Axis is given, provided the Angle made by that Diameter, and its Ordinates is alſo known.

*Cor.* 1. Hence if the vertex Axis and one Point of a Parabola be given, the ſame may be deſcribed: for from thence the Parameter may be found by joining the correſponding Points *s*, *o*, and biſecting the Line *so* perpendicularly; for that will meet the Axis produced in *c*, the Middle betwixt the Points P and *o*.

## Problem II.

*The Tranſverſe and Conjugate Axis of an Ellipſe being given, to deſcribe the ſame* in Plano.

Draw the two indefinite Lines AB, CD, (*Fig.* 27.) perpendicular to each other, in the latter of which

which aſſume any Point V for the Vertex of the Ellipſe, and from V on CD ſet VP equal to the Semiconjugate Diameter, and on P with the Diſtance PV deſcribe a Semicircle *s*V*s*; alſo through P draw any Number of Lines P*r*, P*r*, P*r*, &c. cutting the Circumference in as many Points *s*, *s*, *s*, *s*, &c. and the Line AB in the ſame Number of Points *r*, *r*, *r*, *r*; then from O, the Middle of the Tranſverſe, or Center of the Section, draw Lines to the ſeveral Points *r*. Laſtly, If through the ſeveral Points *s*, *s*, *s*, &c. Lines are drawn parallel to VO*v* the Axis, they'll cut the correſponding Lines O,*r*, O,*r*, O,*r*, &c. in the Points *m*, *m*, *m*, *m*, &c. which are the Points of the Ellipſe ſought: Or if MM be the conjugate Axis, and MR is drawn perpendicular to it, let MP be taken equal to the Semitranſverſe, and a Circle deſcribed thereon, with the Diſtance PM, then proceed as above to find the Points *m*, *m*, *m*, &c. And this is not only a very ready Method for denominating any Ellipſe when its Axes are given, but extends to thoſe Caſes when any two conjugate Diameters are given, and the Angle they make with each other. The Reaſon of this Conſtruction is eaſy, for thro' any of the Points *s*, and its correſponding *m*, draw Lines parallel to AB, let the firſt meet CD in *q*, and the ſecond in *o*; call the tranſverſe *t*, conjugate *c* and put V*o*$=x$, alſo *mo*$=sq=y$, then P*q*$=\frac{\sqrt{c^2-4y^2}}{2}$, and the ſimilar Triangles P*sq*, P*r*V gives $\frac{\sqrt{c^2-4y^2}}{2} : y :: \frac{c}{2} : \frac{cy}{\sqrt{c^2-4y^2}} =$ V*r*; alſo from the ſimilar Triangles O*om*, and OV*r*, we have $\frac{t}{2} - x :$ $y :: \frac{t}{2} : \frac{ty}{t-2x} =$ V*r*, hence $\frac{ty}{t-2x} = \frac{cy}{\sqrt{c^2-4y^2}}$ or $\frac{t}{t-2x}$

$\frac{t}{t-2x} = \frac{c}{\sqrt{c^2-4y^2}}$, or $t^2 \times c^2 - 4y^2 = t-2x^2 \times c^2$, that is $t^2c^2 - 4t^2y^2 = t^2c^2 - 4tc^2x + 4c^2x^2$, whence $y^2 = \frac{c^2}{t^2} \times tx - x^2$, which is the Equation of the Ellipſe, (*Cor.* 2. *Prop.* 1.) therefore the Curve deſcribed is an Ellipſe whoſe tranſverſe is $t$, and conjugate $c$. Q. E. F.

## Problem III.

*The Parameter and Tranſverſe Axis of an Hyperbola being given, to deſcribe the ſame* in Plano.

Draw the Lines AB, CD, (*Fig.* 28.) cutting each other perpendicularly in the Point V, taken for the Vertex of the Hyperbola; then on CD take VC equal to half the tranſverſe Axis, and on the contrary Side AB, take VP equal to half the Parameter; on CP as a Diameter deſcribe a Circle CRPR cutting AB in the Points R, R, through which, and C, draw two indefinite right Lines CR*r*, CR*r*, and through the Points V, let any Number of right Lines be drawn, cutting the Lines CR, CR, in the Points *r*, *r*, *r*, *r*, &c. and *s*, *s*, *s*, *s*, &c. Thence if from the ſeveral points *r* the correſponding Diſtances V*s*, V*s*, V*s*, V*s*, &c. be laid along the Lines *r*V, *r*V, *r*V, *r*V, &c. you will have the Points *m*, *m*, *m*, *m*, &c. of the Hyperbola ſought.

For through any point *s* and its correſponding *r*, and *m*, draw the lines *sq*, *rw*, and *mo*, as alſo *mn* parallel V*w*; then becauſe V*s*=*rm* by conſtruction, and the Angle *s*V*q*=*r*V*w*, the Lines *rn* and *mn* will be reſpectively equal to *sq* and *q*V, for the Angles at *n* and *q* are Right. Now 'tis manifeſt the Triangles CVR and C*wr* are equiangular,

angular, whence if the tranſverſe is called $t$, Parameter $p$ the Line RR, $c$ and V$o=x$; alſo $mo=y$ and $q$V$=mn=e$, we have $t : c :: \frac{t}{2}+x+e : rw$; therefore $rw=\frac{c}{2t}\times\overline{t+2e+2x}$; but the equiangular Triangles $Cqs$ and CVR gives $t : c :: \frac{t-2e}{2} : qs$, therefore $qs=\frac{c}{2t}\times\overline{t-2e}=rn$, whence $mo=y=\frac{c}{t}\times\overline{2e+x}$, but the Triangles V$sq$ and V$om$ are equiangular; thence $e : \frac{c}{2t}\times\overline{t-2e} :: x : y$, therefore $ey=\frac{cx}{2t}\times\overline{t-2e}$; from this Equation we have $\overline{2e=\frac{ctx}{ty+cx}}$, and from the former Equation $2e=\frac{ty-cx}{c}$, therefore $\frac{ty-cx}{c}=\frac{ctx}{ty+cx}$, or $c^2tx=t^2y^2-c^2x^2$; therefore $y^2=\frac{c^2}{t^2}\times\overline{tx+xx}$, but it is evident from the Conſtruction $c^2=tp$, therefore $y^2=\frac{p}{t}\times\overline{tx+x^2}$, which is the Equation of the Hyperbola, (*per Cor. Prop.* 3.) therefore the Curve deſcribed is an Hyperbola, whoſe tranſverſe Axis is $t$, and Parameter $p$. Q. E. F.

By this Method a Parabola might be deſcribed, when any Diameter and its Parameter are given together with the Angle form'd by the ſaid Diameter and its Ordinates. But what has been ſaid is ſufficient for our purpoſe, therefore now we ſhall proceed to ſhew how from theſe Propoſitions, and the following Lemma, thoſe Spaces may be meaſured which are included by any of the Conic Sections.

## CHAP. III.

### *Of the Method of meaſuring Curve-lined Spaces, with the Meaſures of the Circle and its Segments, as alſo the Conic Sections.*

IN the firſt Chapter we have ſhewn the Method of meaſuring all kinds of right-lined Spaces, which originally depends on the *fourth Propoſition* of the *firſt Book of* Euclid, where is inferred the Equality of two Triangles by their coinciding in all their Parts when mutually applied to each other.

But ſince all Figures, of what kind ſoever, are meaſured by the Ratio which they have to the meaſuring Unit, (See *Defin.* 23. *Ch.* 1.) a Figure contained wholly by right Lines, it follows from that Principle only, the Meaſures of Figures contained by Curves cannot be had; becauſe no right-lined Figure, tho' equal in Area to a curve-lined one, can exactly correſpond with it in all its Parts, and therefore ſome other Principle beſides that given for right-lined Figures is neceſſary. An Expedient formerly brought for this purpoſe (by *Archimedes* under the Name of the Method of *Exhauſtions*, ſince greatly improved by *Cavallerus*, and Dr. *Wallis*) was to ſuppoſe the given Figure, whether Plane or Solid, divided into an infinite Number of Parts by Lines or Planes, either all parallel to one another, or all iſſuing from a given Point (as in meaſuring the ſeveral kinds of Spirals, *&c.*) theſe Parts may properly be called the *Elementa* of the Figures; then theſe Elementa being very ſmall, were meaſurable from the Principles delivered already for right-lined Figures, becauſe, tho' contain'd partly by right Lines, and partly by Curves, yet the curved Part being very ſmall, it may be eſteem'd a ſtreight Line; and the Elementa

menta it ſelf equal to another ſimilar to it, which is wholly contain'd by right Lines, or (if a Solid) plane Surfaces; whence after this, nothing further was wanted, but a Method of ſumming theſe Elementa, or their Meaſures. Which, however difficult it may appear at firſt Sight, *viz.* to ſum a Series conſiſting of an infinite Number of Terms, yet in many Caſes 'tis an eaſier Problem, than when their Number is finite.

The moſt Illuſtrious *Newton* conſider'd this Matter after a different manner; for inſtead of the Elementa themſelves, he ſubſtituted the Velocities by which they may be ſuppoſed generated. But this is a Conſideration too far fetch'd for Beginners in theſe Speculations, I ſhall therefore confine my ſelf to the former, and give the Principles and Uſes thereof in as clear and general a Method, as the Nature of the Thing ſeems to admit of. In order to which, we premiſe this

## LEMMA.

*Let* $\overline{o}^m$, $\overline{a}^m$, $\overline{2a}^m$, $\overline{3a}^m$, $\overline{4a}^m$, *&c.* . . . . $\overline{na}^m$ *be an infinite Series of the* m *Powers of Quantities in arithmetic Progreſſion from* o *to* na *incluſive, where the Number of ſignificant Terms is the infinite Number* n; *then ſhall* (s) *the Sum of them all be expreſs'd by* $\frac{\overline{a}^m}{m+1} \times \overline{n}^{m+1} = \frac{n}{m+1} \times \overline{an}^m$, *or putting the laſt Term* l, *viz.* $\overline{an}^m = l$, then $s = \frac{nxl}{m+1}$.

## DEMONSTRATION.

For if $s = \frac{a^m}{m+1} \times n^{m+1}$, when the Number of Terms is *n*, that when this is diminiſhed by *l*, (the

laſt Term) or the Number of Terms is $n-1$, ſhall be $s-l = \frac{am}{m+1} \times \overline{n-1}^{m+1}$, *(viz.)* by writing $s-l$ for $s$, and $n-1$ for $n$; but $\overline{n-1}^{m+1} = n^{m+1} - \frac{m+1}{1} n^{m} + \frac{m+1}{1} \cdot \frac{m+0}{2} n^{m-1} - \frac{m+1}{1} \cdot \frac{m+0}{2} \cdot \frac{m-1}{3} n^{m-2}$ &c. therefore $v-l = \frac{a^m}{m+1} \times \overline{n^{m+1} - \frac{m+1}{1} \cdot n^{m} + \frac{m+1}{1} \cdot \frac{m+0}{2} \cdot n^{m-1}}$ — &c. Let this be taken from $s = \frac{a^m}{m+1} \times \overline{n}^{m+1}$, and there remains $l = \frac{a^m}{m+1} \times \overline{\frac{m+1}{1} \cdot n^{m} - \frac{m+1}{1} \cdot \frac{m+0}{2} \cdot n^{m-1} + \frac{m+1}{1} \cdot \frac{m+0}{2} \cdot \frac{m-1}{3} \cdot n^{m-2}}$, &c. But ſince by Suppoſition $n$ is a Number infinitely great, therefore the firſt Term of the above Value of $l$, muſt be infinitely greater than any of the ſubſequent ones; hence all of them, but the firſt, may be neglected, and then we find $l = \frac{a^m}{m+1} \times \frac{m+1}{1} n^m$, or $l = \overline{an}^m$, as it is by ſuppoſition, whence the Sum was rightly aſſigned. *Q. E. D.*

But ſince $l = \overline{an}^m$, let $l$ be put for $\overline{an}^m$ in the above Value of $s$, and we have $s = \frac{n \times l}{m+1}$; *that is the Sum of an infinite Series of* (*the* m) *Powers of Quantities in Arithmetic Progreſſion from* o, *is equal to the Product of the laſt Term, by the Number of Terms, and this divided by the Index* (m) *plus Unity.*

*Scholium.* This Lemma contains the whole of Dr. *Wallis's Arithmetic of Infinites*, which he has eſtabliſh'd by an Inductionary Method; its Uſe in ſquaring of Curves, and cubing Solids, ſhall be explained in the following Propoſitions.

But

But we muſt obſerve from the ſame Principle by which we have given the Sum of a Series of Powers conſiſting of an infinite Number of Terms, we might deduce that when they are finite, and then the former will only be a particular Care of the latter.

For let $p^m + \overline{p+r}^m + \overline{p+2r}^m + \overline{p+2r}^m \ldots \overline{p+nr}^m$ be a Series of Powers, whoſe Roots are in arithmetic Progreſſion, and put

$$A = \frac{r^m}{m+1}$$

$$B = r^{m-1}\,p + \frac{m+1}{2}A$$

$$C = \overline{r^{m-2}\,p^2 - \frac{m+1}{3}A + B \times \frac{m}{2}}$$

$$D = \overline{\frac{m}{3}\cdot r^{m-3}\,p^3 + \frac{m+1}{3}\cdot\frac{m}{4}A - \frac{m}{3}B + C \times \frac{m-1}{2}}$$

$$E = \overline{\frac{m}{3}\cdot\frac{m-1}{4}\,r^{m-4}\,p^4 - \frac{m+1}{3}\cdot\frac{m}{4}\cdot\frac{m-1}{5}\cdot A + \frac{m}{3}\cdot\frac{m-1}{4}\cdot B - \frac{m-1}{3}C + D \times \frac{m-2}{2}}$$

$$F = \overline{\frac{m}{3}\cdot\frac{m-1}{4}\cdot\frac{m-2}{5}\,r^{m-2}\,p^5 + \frac{m+1}{3}\cdot\frac{m}{4}\cdot\frac{m-1}{5}\cdot\frac{m-2}{6}A - \frac{m}{3}\cdot\frac{m-1}{4}\cdot\frac{m-2}{5}\cdot B + \frac{m-1}{3}\cdot\frac{m-2}{4}C - \frac{m-2}{3}D + E \times \frac{m-3}{2}}$$

*&c.*

Then S, the Sum of all the Terms ſhall be expreſs'd by this Theorem, *viz.*

$$S = An^{m+1} + Bn^m + Cn^{m-1} + Dn^{m-2} \ldots + p^m$$

Where it is to be obſerved, the Terms A, B, C, *&c.* muſt be found till you have got that belong-

ing to the Term where the Index of $r$ is 1; and when that is a Fraction, the Series is infinite; But if $p=o$, then theſe Terms are expreſs'd thus, *viz.*

$$A = \frac{r^m}{m+1}$$

$$B = r^m \times \tfrac{1}{2}$$

$$C = r^m \times \frac{m}{3 \cdot 4}$$

$$D = o$$

$$E = -r^m \times \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

$$F = o$$

$$G = r^m \times \frac{m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \overline{m-3} \cdot \overline{m-4}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}$$

*&c.* *&c.*

As found by my ingenious Friend Mr. *Tho. Sympſon*, in a late Treatiſe on the Laws of *Chance*. But the above Theorem for ſumming of Series of Powers, has lain by me ſome Years, nor ſhould I ever have made it publick, had not this Occaſion offered, becauſe the differential Method of Sir *Iſaac Newton* is better fitted for ſumming Series's, and which has been largely inſiſted on by Mr. *James Stirling*, in a whole Treatiſe on that Subject.

## Prop. I.

*The Meaſure of every Sector of a Circle is equal to half the Product of the Meaſure of a Radius thereof, into the Arch which is the Baſe of the Sector.*

For let CAE (*Fig.* 29.) be the Sector whoſe Center is C, and Baſe the Arch AE, which ſuppoſe divided into an infinite Number of Parts in the Points *a*, *b*, *c*, *d*, *&c.* and draw the Chord A*a*,

on

on which let fall the Perpendicular CP, which will be infinitely near equal to CA, the Radius of the Sector: put $r$ for the Meaſure of CA, and $q$ the Meaſure of (A$a$=$ab$=$ac$=$ad$, &c.) one of the Parts into which the Baſe of the whole Sector is divided, and $l$ the Meaſure of the Baſe of the Sector, *viz.* $l$=AE, then $\frac{l}{q}$ is the Number of the Elementa CA$a$, C$ab$, C$bc$, C$cd$, &c. (for $q : 1 :: l : \frac{l}{q}$) but 'tis evident they are all equal, hence CA$a \times \frac{l}{q}$ is the Meaſure of the whole Sector, but the Meaſure of CA$a$ is $\frac{r \times q}{2}$ (*Cor.* 2. *Prop.* 1. *Chap.* 1.) hence the Meaſure of the whole Sector is $\frac{r \times q}{2} \times \frac{l}{q} = \frac{r \times l}{2}$. *Q. E. O.*

*Corol.* 1. If on C with any other Radius as R an Arch of a Circle be deſcribed, cutting CA, CE in in $\alpha$, $\varepsilon$, then the Area of the latter Sector ſhall be to that of the former, as $R^2$ : is to $r^2$ ; for $\frac{\alpha\varepsilon \times R}{2}$ is the Area of the Sector C$\alpha\varepsilon$, and $r : l :: R : \frac{lR}{r}$ $=\alpha\varepsilon$, per Figure; therefore $\frac{l \times R^2}{2R}$ is the Area of the latter, and $\frac{r \times l}{2}$ is the Area of the former: but $\frac{l \times R^2}{2r} : \frac{l \times r}{2} :: R^2 : r^2$. *Q. E. D.*

*Cor.* 2. Whence the Meaſure of every Circle is equal to the Product of a Radius thereof into half the Circumference. Therefore before the Meaſure of a Circle can be had, we muſt ſhew how to find its Circumference from the Diameter being given.

PROP.

## Prop. II.

*Let* t *be the Tangent of* BE (Fig. 30.) *any Arch of a Circle (not more than* 45 *Degrees) whoſe Center is* C, r *the Radius thereof; then ſhall the Arch* BE *be expreſſed by the following Series,* $t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}}$, &c. *infinitely.*

For let the Tangent BT be conceived divided into an infinite Number of equal Parts in the Points $a$, $a'$, $a''$, $a'''$, &c. to which from the Center let right Lines be drawn and produced, cutting the Arch in the Points $e$, $e'$, $e''$, $e'''$, $e''''$, &c. alſo on the Center C with the ſeveral Diſtances $Ca$, $Ca'$, $Ca''$, $Ca'''$, &c. deſcribe the Arches $Tv$, $vv'$, $vv''$, &c. cutting the produced Lines in as many Points $v$, $v'$, $v''$, $v'''$, &c. now 'tis evident the Arch $Tv$ is infinitely ſmall, and ſo may be taken for a right Line; alſo the Triangles TBC and $Tva$ may be taken for Equiangular; therefore put $q$ to denote the Meaſure of any are of the equal Parts $Ta$, $aa'$, $aa''$, $aa'''$, &c, then $\frac{t}{q}$ is their Number. Now from the 4th *Euclid* 6. we have CT : CB : : $Ta$ : $Tv$, that is $\sqrt{r^2+t^2} : r :: q : \frac{rq}{\sqrt{r^2+t^2}} = Tv$, but the Triangles $Cv$T, and $CEe$ are equiangular; Therefore CT : $Tv$ : : CE : $Ee$, that is $\sqrt{r^2+t^2} : \frac{rq}{\sqrt{r^2+t^2}} :: r : \frac{r^2q}{r^2+t^2} = Ee$, therefore by Diviſion $Ee = q \times t - \frac{t^3}{r^2} + \frac{t^5}{r^4} - \frac{t^7}{r^6} + \frac{t^9}{r^8}$, &c. See the Proceſs

$r^2+t^2)$

$r^2 + t^2) r^2 ( 1 - \frac{t^2}{r^2} + \frac{t^4}{r^4} - \frac{t^6}{r^6}$, &c.

$$\begin{array}{l} r^2+t^2 \\ \hline -t^2 \\ \quad -t^2-\frac{t^4}{r^2} \\ \hline \quad +\frac{t^4}{r^2} \\ \quad \frac{t^4}{r^2}+\frac{t^6}{r^4} \\ \hline \quad -\frac{t^6}{r^4} \\ \quad -\frac{t^6}{r^4}-\frac{t^8}{r^8} \\ \hline \quad +\frac{t^8}{r^8} \end{array}$$

After the same manner calling B*a*, B*a'*, B*a''*, B*a'''*, &c. *t*, *t'*, *t''*, *t'''*, &c. successively, you will find

$$e\,e' = \frac{qr^2}{r^2+{}'t^2} = q \times 1 - \frac{{}'t^2}{r^2} + \frac{{}'t^4}{r^4} - \frac{{}'t^6}{r^6} + \frac{{}'t8}{r^8}\ \&c.$$

$$e'e'' = \frac{qr^2}{r^2+{}''t^2} = q \times 1 - \frac{{}''t^2}{r^2} + \frac{{}''t^4}{r^4} - \frac{{}''t^6}{r^6} + \frac{{}''t^8}{r^8}\ \&c.$$

$$e''e''' = \frac{qr^2}{r^2+t'''^2} = q \times 1 - \frac{{}'''t^4}{r^2} + \frac{{}'''t^4}{r^4} - \frac{{}'''t^6}{r^6} + \frac{{}'''t^8}{r^8}\ \&c.$$

From hence 'tis manifest the Length of the Arch BE = E*e* + *e e'* + *e'e''* + *e''e'''*, &c. is equal to the Sum of the following Series, contained both ways to Infinity.

*viz.*

*viz.* BE =

$$q \times 1 - \frac{t^2}{r^2} + \frac{t^4}{r^4} - \frac{t^6}{r^6} + \frac{t^8}{r^8} - \frac{t^{10}}{r^{10}} \quad \&c.$$

$$q \times 1 - \frac{t'^2}{r^2} + \frac{t'^4}{r^4} - \frac{t'^6}{r^6} + \frac{t'^8}{r^8} - \frac{t'^{10}}{r^{10}} \quad \&c.$$

$$q \times 1 - \frac{t''^2}{r^2} + \frac{t''^4}{r^4} - \frac{t''^6}{r^6} + \frac{t''^8}{r^8} - \frac{t''^{10}}{r^{10}} \quad \&c.$$

$$q \times 1 - \frac{t'''^2}{r^2} + \frac{t'''^4}{r^4} - \frac{t'''^6}{r^6} + \frac{t'''^8}{r^8} - \frac{t'''^{10}}{r^{10}} \quad \&c.$$

*Infinitely.*

*Infinitely.*

Now since the Number of Terms in each vertical Row is $\frac{t}{q}$ by what was said above, therefore the Sum of $\overline{1\times1\times1\times1\times}$, &c. $\times q = \frac{t}{q} \times 1 \times q = t$; and the second is a Series of Squares, whose Roots $t$, $t'$, $t''$, $t'''$, &c. are in arithmetic Progression, the greatest being $\frac{t^2}{r^2}$, hence in the preceding Lemma for $l$,

$n$,

$n$, and $m$, writing $\frac{t^2}{r^2}$, $\frac{t}{q}$ and 2 reſpectively, we have the Sum of $q \times \overline{\frac{t^2}{r^2} + \frac{{}^{I}t^4}{r^4} + \frac{{}^{II}t^6}{r^6}}$, &c. $= q \times \frac{t^2}{r^2} \times \frac{t}{q} \times \frac{1}{2+1} = \frac{t^3}{3r^2}$: After the ſame manner the Sum of the third Row will be $\overline{\frac{t^4}{r^4} + \frac{{}^{I}t^6}{r^6} + \frac{{}^{II}t^8}{r^8}}$ &c. $\times q = q \times \frac{t^4}{r^4} \times \frac{t}{q} \times \frac{t}{4+1} = \frac{t^5}{5r^4}$, the Sum of the fourth Row will be $q \times \frac{t^6}{r^6} \times \frac{t}{q} \times \frac{1}{5+1} = \frac{t^7}{7r^7}$, and ſo on; whence connecting theſe Sums under their proper Signs for the Length of the Arch BE, we have $t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}}$, &c. *Q. E. I.*

It may be obſerved it was not neceſſary in this Proceſs, to find more than the firſt horizontal Row of theſe Series, becauſe each of that being the laſt or greateſt Term in the ſeveral Powers, and the Form of all the reſt the ſame; the whole might have been ſummed from the Lemma without further trouble; but we ſet the Values of the ſeveral Arches $Ee$, $ee'$, $e'e''$, $e''e'''$, &c. down for the ſake of Beginners, becauſe it muſt be more ſatisfactory to them, ſince having exhibited the Values of the ſeveral Elementa, their Sum muſt manifeſtly be equal to the whole Arch.

*Cor.* 1. Now let CB=1, and the Arch BE be 30 *Degr.* then its Tangent is manifeſtly equal to $\sqrt{\frac{1}{3}}$, for the Sine of 30 *Degr.* is $\frac{1}{2}$, *viz.* half the Chord of 60 *Degt.* which is equal to the Radius,

therefore the Coſine is equal to $\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$; but as the Coſine of any Arch is to its Sign, ſo is the Radius of the Tangent thereof, that is $\frac{\sqrt{3}}{2} : \frac{1}{2} :: 1 \sqrt{\frac{1}{3}}$, hence $BE = \frac{1}{\sqrt{3}} - \frac{1}{3.3\sqrt{3}} + \frac{1}{5.9\sqrt{3}} - \frac{1}{7.27\sqrt{3}} + \frac{1}{9.81\sqrt{3}} - \frac{1}{11.243\sqrt{3}}$, *&c.* and ſix times this muſt be the Length of the Semicircumference; therefore put $6\sqrt{\frac{1}{3}}=\sqrt{12}=A$, $\frac{A}{3.3}=B$, $\frac{3B}{5.3}=C$, $\frac{5C}{7.3}=D$, *&c.* Then half the Circumference is equal to A,—B+C—D+E—F, *&c.* See the Proceſs,

$\sqrt{12}$ = A = 3.4641.0161.5
$\frac{B}{5}$ = C = . . 769.8003.6
*&c.* E = . . . 47.5185.4
G = . . . 3.6552.7
I = . . . . . 3105.8
L = . . . . . 279.4
N = . . . . . 26.1
P = . . . . . 2.5
R = . . . . . 2
*&c.*

A+C+E+G+L = 3.5462.3317.2
4046.4051.7

3.1415.9265.5

$\frac{A}{1}$

$\frac{A}{q} = B =$, .3849.0017.9

$\frac{5C}{21} = D =$, .. 183.2858.0

*&c.* $F =$ ... 12.9596.0

$H =$ .... 1.0559.7

$K =$ ..... 926.3

$M =$ ..... 85.0

$O =$ ..... 8.0

$Q =$ ..... 8

*&c.*

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$B + D + F + H = 4046.4051.7$

The latter being taken from the former, leaves 3.14.15.9265.5 for the Length of half the Circumference of a Circle whoſe Radius is Unity: *Therefore the Diameter of any Circle is to its Circumference as* 1 *is to* 3.1415.9265.5 *nearly.*

There are various other Ways of finding the Circumference of a Circle, particularly one given by Mr. *Machin*, and publiſhed in the *Synopſis Palmariorum Matheſeos*, of that incomparable Mathematician Mr. *Jones*, where is ſhewn that the Diameter of a Circle being 1, the Circumference will be $\frac{16}{5} - \frac{4}{239} - \frac{1}{3} \times \frac{16}{5^3} - \frac{4}{239^3} + \frac{1}{5} \times \frac{16}{5^5} - \frac{4}{239^5}$ *&c.* a few Terms of which will exhibit the ſame Number we found above. But my Friend, to whoſe Aſſiſtance I am ſo much indebted in this Performance, has favoured me with a Method by which the Solution above may be much abbreviated: For after any Number of Terms of the above Series is found, if you put $n$, the Index of $t$ in the laſt of them, which we will call T, then the Sum of the remaining Terms ſhall be $+ \frac{n \times T t^2}{n+2 \times r^2 + t^2}$

4

$+\frac{2T't^2}{n+4.r^2+t^2}+\frac{2.2T''t^2}{n+6.r^2+t^2}+\frac{2.3T'''t^2}{n+8.r^2+t^2}$
$+\frac{2.4T''''t^2}{n+10.r^2+t^2}$ + &c. Where we muſt obſerve the Sign of T′ is ever contrary to that of T. the Terms T, T′, T″, T‴, &c. ſucceeding in Order; thus in the preceding Example, the Sum of the firſt ſix Terms, *viz.* $t-\frac{t^3}{3r^2}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}+\frac{t^9}{9r^8}$
$-\frac{t''}{11.r^{10}}\times 6$, is equal to 3.1413.0878.6; and here $n=11$, and the ſixth Term is $\frac{t''}{11r^{10}}=$ ,0012.95960=T; hence youl'll find $\frac{11\times T\times t^2}{13\times r^2+t^2}$
$=\frac{11T}{52}=T'=$ ,0002.74145
$\frac{1.2T'}{60}=T''=\ldots\ldots 9138$
$\frac{2.2T''}{68}=T'''=\ldots\ldots 537$
$\frac{3.2T'''}{76}=T''''=\ldots\ldots 43$
$\frac{4.2T''''}{84}=T'''''=\ldots\ldots 4$

Their Sum + ,0002.83867

To which the firſt ſix Terms being added, *viz.* 3.1413.08786, we have for the Semi-circumference 3.1415.92653 the ſame as was found by 17 Terms of the original Series; whereas by this Method we had but 11, and if more initial Terms had been col-

collected, we had ſtill obtained the Sum of the remaining ones by a leſs Number of theſe in Proportion.

*Cor.* 2. Hence if D be the Diameter of any Circle, and $a$=,7854, then ſince 1 : 3,1416 (=4$a$) : : D : 4$a$D the Circumference of the Circle whoſe Diameter is D, and therefore (by *Cor.* to the laſt *Propoſit.*) $\frac{D}{2} \times \frac{4aD}{2} = aD^2$ is the Area thereof.

Whence we have the following *Rule to meaſure any Circle whoſe Diameter is given.* Multiply the Square of the Diameter by ,7854, and the Product is nearly the Area thereof.

*Cor.* 3. If C be the Circumference of any Circle, then ſince 3.1416=4$a$ : 1 : : C : $\frac{C}{4a}$ the Diameter thereof; therefore by the ſame *Corol.* $\frac{C}{2} \times \frac{C}{8a} = \frac{C^2}{16a}$ is the Area thereof, which gives this *Rule for Meaſuring any Circle whoſe Circumference is given.*

Divide the Square of the Circumference by 12.5664, and the Quotient is (nearly) the Area thereof. And from hence we have the two following *Rules to meaſure any Circle by the Sliding-Rule.*

*Rule* 1. Set 1 on D to ,7854 on C, then oppoſite the Diameter on D is the Area on C.

*Rule* 2. Set 1 on D to ,079578 on C, then oppoſite the Circumference on D is the Area on C.

Or ſet 3.5449 on D to 1 on C, then oppoſite the Circumference on D is the Area on C.

*Examp.* Let the Diameter of any Circle be 20, and its Area required, the Square of the Diameter is - - - - 20x20=400
Which multiply'd - by 0,7854
And the Product is - 314,1600 the Area thereof.

*By the Sliding-Rule*, ſet 1 on D to ,7854 on C, then oppoſite 2 on D (for 20 is off the Rule) is 3.1416 on C; and becauſe 20 was divided by 10, I multiply 3,1416 by 100, the Square of 10, and find 314.16 for the Area as before. (See *Prop.* 5. *Sect.* 2. *Part* I.)

But if 20 be multiplied by 3.1416, we have 62.832 for the Circumference; hence by the ſecond Rule above, I ſet 1 on D to ,079 on C, then oppoſite 3.14 (*viz.* the $\frac{1}{20}$ of 62,832) on D, is ,7854 on C; which I multiply by 400, the Square of what the Circumference was divided by, and find 314,16 for the Area as before. Or by the ſecond Method 3,5449 on D ſet to one on C, then againſt 6.2832 on D is 3.1416 on C, which multiply'd by 100 (for a manifeſt Reaſon) gives 314,16 as before.

*Corol.* 4. Hence if 180 Degrees is equal in Length to 3.1415926 *&c.* ſo is one Degree to $\frac{3,1415926}{180} = ,01745329$, the Length of one Degree in ſuch Parts as the Radius is 1; therefore $,01745329 \times r$ is the Length of one Degree when the Radius is $r$; conſequently the Length of $d$ *Degrees* is $,01745329 \times rd$, and this multiplied by $\frac{r}{2}$ gives $,008726647 \times r^2 \times d$ for the Area of a Sector whoſe Radius is $r$, and the Baſe conſiſts of $d$ *Degrees*; which may be expreſſed thus,

*The Number of Degrees in any Sector multiplied by* ,008726647, *and that Product by the Square of its Radius, ſo ſhall this laſt Product be the Area of the Sector.*

*Or*, if Unity be divided by 008726647, and the Square Root of the Quotient be extracted, it will be found 10,7048; therefore if 10,7048 on D is ſet to the Degrees in the Sector's Baſe on C, then

then oppoſite the Radius (of the Sector) on D, is the Content on C.

*Examp.* Let the Radius of the Sector be 62.5, and Baſe 80 *Degr.* then $62.5 \times 62.5 \times 80 \times 0087266 47 = 2727,08$ nearly the Area. Or 10.7048 on D ſet to $\frac{80}{4}$ on C, then oppoſite 62.5 on D is 681.77 on C; and this multiplied by 4, becauſe 80 was divided thereby, gives 2727.08, the ſame as before. (See *Prop.* 5. *Sect.* 2. *Part.* 1.)

*Cor.* 5. From the laſt *Corol.* is deduced the Rule for meaſuring the Segment of a Circle; for if from the Area of the Sector, be taken the Area of a Triangle, two of whoſe Sides are its Radii, and the third the Chord of its Baſe, there will remain the Area of the Segment; now put $q = ,0174532$, and $s$ the Sine of $d$ *Degrees* in the Table of *Natural Sines*, then $r \times s$ is its Sine when the Radius is $r$, hence $rs \times \frac{r}{2} = \frac{s \times r^2}{2}$ is the Area of the ſaid Triangle; therefore $\frac{qdr^2 - r^2 s}{2}$ is the Area of the Segment, which gives the following Rule to meaſure any Segment of a Circle.

*Let the Number of Degrees in the Segment be multiplied by* ,0174533 *and the natural Sine of thoſe Degrees (from the Tables) taken from the Product; then half the Remainder multiplied by the Square of the Radius of the Segment is the Area thereof.*

*Examp.* Let us reaſſume the laſt Example, where the Arch was 80 Degrees: its Sine in the Tables is 9848077, and 80 multiplied by 0174533 is 1,3962560, from which ,9848077 being taken, there remains ,411483; half this multiplied by $62.5 \times 62.5 = 3906,25$ gives 803,61, nearly the Area of the Segment.

This may be done at two Operations by the Sliding-Rule; for let the Area of the Sector be

found by the laſt *Corol.* then ſet 1 on D to half the Sine of the Segment's Arch from the Tables on C, and oppoſite the Radius on D is the Area of the Triangle on C, ſo in this Example 1 on D ſet to ,4924038 againſt $\frac{625}{200}=3.125$ on D (for 62.5 is off the Rule) is 4,8086, which multiplied by 400 the Square of the Diviſor of 62.5 gives 1923,44; this taken from 27.27, the Sector's Area, leaves 803,64 for the Area of the Segment, nearly the ſame as before. But there ſeems ſtill wanting a Method of computing the Area of any Segment of a Circle without the Arch, that being ſeldom a Part of the Data in Practice, therefore we ſhall now lay down a *New Method* for that Purpoſe; for the Reader by purſuing the ſame Steps with thoſe for obtaining the Arch from its Tangent, might get the common Series for the Segment; but in place of that Series, which is to be met with in moſt Authors that have wrote on the Quadratures of Curves, I ſhall preſent the Reader with a new one, both converging *faſter* and more *ſimple*.

### Prop. III.

Let C (*Fig.* 31.) be the Center of the Segment EBDE, whoſe verſed Sine is BG $= v$, the Radius CB $= \frac{d}{2}$, and V $= d - v$, then the Area of the Segment EBDE ſhall be expreſſed by this Series,

$$4\sqrt{vV}\times\frac{v}{1\cdot3}+\frac{v^2}{1\cdot3\cdot5V}-\frac{v^3}{3\cdot5\cdot7V^2}+\frac{v^4}{5\cdot7\cdot9V^3}-\frac{v^5}{7\cdot9\cdot11V^4}$$

&c. or put A for the firſt Term, *viz.* $A=\frac{4v}{3}\sqrt{vV}$, B the ſecond, C the third, then the Area of the Segment is $A+\frac{Av}{5V}-\frac{1Bv}{7V}+\frac{3Cv}{9V}-\frac{5Dv}{11V}+\frac{7Ev}{13V}-$&c.

The

The Demonſtration of this Series depends on Principles not yet publickly known, ſo the Reader muſt take it on truſt till my Friend ſhall think proper to publiſh not only that, but ſeveral other uſeful Improvements he has made in this Buſineſs of Quadratures, as well as ſeveral things in the Method of Infinite Series, and the other recluſe Parts of Mathematicks.

But we muſt not paſs over that uſeful Approximation given for the Area of the Segment of a Circle by the Great *Newton*, and publiſh'd by Dr. *Wallis*, and the moſt accurate Mr. *Jones*; and alſo by Mr. *Colſon*, in his late Verſion of *Newton's Fluxions*, in the laſt of which is its Demonſtration; the Theorem is this: If the Diameter of any Circle be $d$, the verſed Sine of any Arch $v$, then the Area of a Segment contain'd by twice the ſaid Arch, and twice its Sine, will be $\sqrt{dv} + 4\sqrt{dv - \frac{3}{4}v^2} \times \frac{4v}{15}$, which may be expreſſed in Words thus; Let the Diameter be multiplied by the verſed Sine of any Segment, and the Product called the firſt Number, from which ſubſtract three fourths of the Square of the ſaid verſed Sine, and call the Remainder the ſecond Number, then the Square Root of the firſt Number added to four times the Square Root of the ſecond, and that Sum multiply'd by four fifteenths of the verſed Sine, will be nearly the Area of the Segment ſought. Or we may give this Approximation for the Area of the Segment of a Circle which in moſt Caſes is nearer than that above; *viz.* $\frac{80d - 39v}{16d - 3v}\sqrt{dv} \times \frac{4v}{15}$ is the Meaſure of the Segment of the Circle, whoſe Diameter is $d$, and verſed Sine $v$.

We will illuſtrate both theſe Methods by an Example: Let the Diameter of any Circle be 50

Inches, and the verſed Sine of any Arch thereof 10 Inches, then $\frac{v}{V} = \frac{10}{40} = \frac{1}{4}$, alſo $\frac{4v\sqrt{vV}}{3} = \frac{4\times10\sqrt{400}}{3} = \frac{800}{3} = 266,6666$ &c. $= A$ alſo $B = \frac{A}{4\cdot5}$, $C = \frac{B}{4\cdot7}$, $D = \frac{3C}{4\cdot9}$, $E = \frac{5D}{4\cdot11}$, $F = \frac{7E}{4\cdot13}$, &c. See the Operation.

| | | | |
|---|---|---|---|
| A = | 266,66667 | C = | 47619 |
| B = | 13,33333 | E = | 451 |
| D = | 3968 | G = | 9 |
| F = | 61 | | |
| | | | — .48079 |
| + | 280.04029 | | |
| — | 48079 | | |
| | 279.55950 the Area of the Segment. | | |

Now if we make uſe of *Newton*'s Rule, $\sqrt{dv} = \sqrt{500} = 22.36069$, and $4\sqrt{dv - \frac{3}{4}v^2} = 4\sqrt{425} = 82,46212$, the Sum of theſe is 104.82281, and this multiplied by $\frac{4v}{15} = \frac{8}{3}$ gives 279,52749; nearly the ſame as before.

Or by the ſecond Approximation $\frac{4000-30}{800-30}\sqrt{500}\times\frac{40}{15} = \frac{3961}{770}\sqrt{500}\times\frac{8}{3} = 279,557$; differing from the truth only ,002. which is leſs than the $\frac{1}{139500}$th of the whole Segment.——And this Method is alſo eaſier in practice.

By any of theſe Methods the Area of a circular Segment may be found, but in Practice a Table of the Areas of the Segments of a Circle, whoſe Diameter is divided into 1000 equal Parts is preferable; therefore ſince ſuch a Table is ſo very uſeful (as will appear farther on) in finding the Quan-

Quantity of Liquor drawn out, or remaining in a Caſk not full, we ſhall proceed to ſhew the manner of computing it by this Propoſition.

Let the Diameter of a Circle be 1, which ſuppoſe divided into 1000 equal Parts, thro' every one of which imagine Perpendiculars drawn and continued both ways to the Circumference.

Then ſince the firſt verſed Sine is ,001, we have $v$=,001 and $\frac{v}{V} = \frac{,001}{999} = \frac{1}{999}$; therefore A= $\frac{4}{3}\frac{v}{}\sqrt{v\overline{V}} = \frac{4}{3000}\sqrt{,000999}$=00004214, and B = $\frac{Av}{5V} = \frac{00004215}{999\times5}$ = 00000001, whence the firſt Number of the Table is ,00004215.

The ſecond verſed Sine is ,002=$v$, therefore V=,998 and $\frac{v}{V} = \frac{2}{998} = \frac{1}{199}$, whence A = $\frac{8}{3000}\sqrt{,001996}$ = ,00011914, and B = $\frac{Av}{5V}$ = =$\frac{00011914}{499\times5}$,00000005; therefore the ſecond Number is ,00011919.

The third verſed Sine is ,003=$v$, therefore V=1—,003 = ,997, whence $\frac{v}{V} = \frac{3}{997}$, and A= $\frac{12}{3000}\sqrt{002991}$ = ,00021876: alſo B = $\frac{Av}{5V}$ = $\frac{,00021876\times3}{5\times997}$ = 00000013, thence the third Number of the Table is ,00021889.

And by this Method the whole might be computed, but after a ſufficient Number of Terms are found at the beginning of the Table, the reſt may be had (to 7 or 8 Places of Decimals) by this

Rule: Let $\alpha,\beta,\gamma,\delta$, denote any four Terms ſucceeding in the Order of the Letters, then $\delta = \alpha + 3\times\overline{\gamma-\beta}$.

And if any Term of this Table be divided by 78539816, or multiplied by its Reciprocal, you'll have the common Table of Segments when the Area is Unity; but the firſt Form is preferable, as will appear further on.

But to ſhew the Uſe thereof, we muſt lay down this

THEOREM.

*If two Circles have the ſame Center, and from thence be drawn two Lines to any two Points in the outer Circumference, thereby cutting the inner one in the ſame Manner, and the Chords of thoſe Arches be drawn, forming a Segment in each; then ſhall the verſed Sines of theſe Segments be to one another, as their reſpective Radii; and the Segments as the Squares thereof.*

Let C (*Fig.* 31.) be the Center of the outermoſt Circle, and E, D, the two Points taken in its Circumference, which are join'd by the Lines CE, CD; alſo let *e b d* be part of another Circle deſcribed on the ſame Center, which cuts the Lines CE, CD in the Points *e*, *d*; draw the Lines ED, *e d*, and the Line CB to B. a Point in the middle betwixt the Points E, D; let it cut the Arch *e d* in *b*, and the Lines ED, *e d* in G, *g* reſpectively; then I ſay GB : *g b*, ſo is CD to *c d*; alſo EBDE : *e b d e* :: $\overline{CD}^2$ : $\overline{Cd}^2$. For in the firſt place CD : CG :: C*d* : C*g*, that is CD : CD—GB :: C*d* : C*d* —*bg*, therefore (by 17 *Eucl.* 5.) CD : GB :: *cd* : *gb*. Again, let us put S for the Sector CDE, *s* the Sector C*de*, alſo T the Segment DBED, and *t* the Segment *d b e d*, then (by the 19 *Euclid* 6.) $\overline{CD}^2$ : $\overline{Cd}^2$ :: S—T : *s*—*t*, but (*Cor.* 1. *Prop.* 1.) S : *s* :: $\overline{CD}^2$ :

$\overline{CD}^2 : \overline{Cd}^2$, therefore S : *s* : : S—T : *s*—*t*, or (as above) S : T : : *s* : *t*, whence $\overline{CD}^2 : \overline{Cd}^2$ : : T : *t*. *Q. E. D.*

*Corol.* From this Theorem and the foregoing Table, we may eaſily compute the Area of any Segment from its verſed Sine being given.

For if *v* be put for any verſed Sine of a Circle whoſe Diameter is *d*, then by the firſt Part of this Theorem *d* : *v* :: 1 : $\frac{v}{d}$ and 1 : 1000 : : $\frac{v}{d}$ : $\frac{1000v}{d}$. Let this be look'd for in the Table of Segments under VS, whoſe Diameter is 1, and call the Number againſt it *n*, then by the ſecond Part of the Theorem $\overline{1}^2 : n : : d^2 : d^2 \times n$, which is the Area of a ſimilar Segment, whoſe verſed Sine is *v*, and Diameter *d*.

We may expreſs this Method in Words thus: Let three Cyphers be annexed to the given verſed Sine, and then divided by the Diameter of its Circle, ſeek the Quotient under VS, and take out the Number againſt it under the Letters *Seg.* Multiply this Number by the Square of the Diameter, and the Product is the Area of the Segment ſought. I am afraid the Reader will think I have taken too much Pains about the Circle, its Sectors and Segments; but they are of ſuch particular Uſe in Gauging, and ullaging of Caſks, that I was willing to ſet this Affair in as clear a Light as poſſible, therefore I hope that will excuſe my ſeeming Prolixity.

## PROP. IV.

*The Area of every Conic Parabola, is equal to two thirds of the Product of the Meaſures of its Abſciſſa and double Ordinate.*

For let MVN (*Fig.* 32.) be a Conic Parabola, whoſe Vertex is V, Axis VO, and MON the double

double Ordinate of the Abſciſſa VO, conceive the Abſciſſa VO divided into an infinite Number of Parts in the Points $o$, $o'$, $o''$, $o'''$, &c. and put the Meaſure of one of them $q$, the Meaſure of V$o$, $x$; and MN, $y$; alſo let the ſucceeding Abſciſſas V$o$, V$o'$, V$o''$, V$o'''$, &c. be $x'$, $x''$, $x'''$, $x''''$, &c. their correſponding Ordinates $y'$, $y''$, $y'''$, $y''''$, &c. and the Parameter of the Parabola $p$: Now ſince V$o$=$x$ is divided into an infinite Number of Parts, each of which is expreſſed by $q$, therefore $\frac{x}{q}$ is their Number: from *Prop.* 2. *Chap.* II. $4px=y^2$, therefore $y=2\sqrt{px}$ and $q\times y=2q\times\sqrt{px}$ the Meaſure of the firſt and greateſt Ellementa M$mn$N, after the ſame manner the ſecond will be $2q\times\sqrt{px'}$, the third $2q\times\sqrt{px''}$, &c. as here under, where for Brevity of Expreſſion we denote the ſeveral Elementa in order by $e$, $e'$, $e''$, $e'''$, &c.

$$\text{Hence}\begin{cases} e=2q\sqrt{px}=2q\sqrt{p}\times\sqrt{x} \\ e'=2q\sqrt{px'}=2q\sqrt{p}\times\sqrt{x'} \\ e''=2q\sqrt{px''}=2q\sqrt{p}\times\sqrt{x''} \\ e'''=2q\sqrt{px'''}=2q\sqrt{p}\times\sqrt{x'''} \\ \text{\&c.} \qquad\qquad \text{\&c.} \end{cases}$$

Therefore the Area of the Parabola is equal to a Series of the $\frac{1}{2}$ Powers of Quantities ($x$, $x'$, $x''$, $x'''$, &c.) in Arithmetic Progreſſion from $o$, multiplied by $2q\sqrt{p}$, hence by the Lemma $2q\sqrt{p}\times x^{\frac{1}{2}}\times\frac{2}{3}\times\frac{x}{q}=\frac{4\sqrt{p}\times x^{\frac{3}{2}}}{3}$ is the Area; for the laſt or greateſt Term is $2q\sqrt{x}$, the Number of Terms ($n$) is $\frac{x}{q}$, and the Index ($m$) is $\frac{1}{2}$, hence $\frac{nl}{m+1}$ in the Lemma gives $\frac{x}{q}\times 2q\sqrt{px}\times\frac{2}{3}=\frac{4x\sqrt{px}}{3}$ but $\sqrt{px}=\frac{y}{2}$, there-

therefore $\frac{2x \times y}{3}$ is the Area of the Parabola. *Q. E. D.*

And from hence by the Sliding-Rule we have this Solution:

Set $1\frac{1}{2}$ on A to the Abſciſſa on B, then oppoſite the double Ordinate on A is the Area on B.

*Examp.* Let the Abſciſſa VO be 124 Inches, the double Ordinate MN=135 Inches, then $\frac{2 \times 135 \times 124}{3}$ =11160 Square Inches the Area.

Or, ſet $1\frac{1}{2}$ on A to 12,4 (the $\frac{1}{10}$ Part of 124) on B, then oppoſite 13.5 (the 10th Part of 135) on A, is 111.6 on B: And becauſe the Terms 135, 124 were each divided by 10, multiply 111,6 by 100 (the Product of their Diviſors) and find 11160 for the Area as before.

## PROP. V.

*If on the Tranſverſe of an Ellipſe a Circle be deſcribed, and any Point taken in the Axis, and a Perpendicular drawn thro' the ſame, cutting of a circular and elliptic Segment; then ſhall the Area of the former Segment be to the latter, as the tranſverſe Axis is to the conjugate.*

For (*Fig.* 33.) let VG*v*G be the Ellipſe, whoſe tranſverſe is V*v*, and conjugate GG, cutting each other in C, the Center of the Ellipſe, and let VM*v*M be a Circle deſcribed on C, with the Diſtance C*v*=CV; alſo take O any Point in V*v*, through which let a Line be drawn perpendicular to the Ax*e*, meeting the circumſcribing Circle in N, N, and the Ellipſe in *n*, *n*, then will the circular Segment N *v* N be to the elliptic Segment *n v n* as V*v* is to GG. For conceive O*v* to be divided into an infinite Number of equal Parts in the Points *o*, *o*, *o*, *o*, &c. thro' which draw Lines parallel

rallel to NN, cutting the circumſcribed Circle in N′, N″, N‴, &c. and the elliptic Curve in *n′*, *n″*, *n‴*, &c. alſo put *q* to denote the Meaſure of one of the Parts O*o*=*oo*=*oo*, now by the Property of the Ellipſe (*Corol.* 4. *Prop.* 1. *Chap.*2.) NN : *nn* : : V*v* : GG ; therefore NN × *q* : *nn* × *q* : : V*v* : GG ; that is the circular Elementa N N N′ N′, is to the elliptic Elementa *n n n′ n′*: as the tranſverſe Axis is to the conjugate ; and the ſame may be ſhewn (after this manner) of all the other Elementa ; wherefore (by 12 *Euclid* 5.) V*v* : GG : : N*v*N : *n v n*. *Q. E. D.*

*Cor.* 1. Hence *the Area of every Ellipſe is to the Area of its circumſcribed Circle, as the conjugate Axis is to the tranſverſe :* Therefore put *a* to denote the Area of the Circle, whoſe Diameter is Unity, then $\overline{Vv}^{2} \times a$, is the Area of the circumſcribed Circle, (*Cor.* 2. *Prop.* 2.) therefore by this Propoſition V*v*: GG : : $\overline{Vv}^{2} \times a$ : GG × *v*V × *a* the Area of the Ellipſe, which gives this Rule: *The Area of every Ellipſe is equal to the Product of the Tranſverſe by the conjugate Axis, and that Product by* .7854. *But for the Sliding-Rule* ; Let us find the Value of $,\frac{1}{7854}$, which is 1.27324, then ſet 1,27324 on B to the tranſverſe on A, and oppoſite the Conjugate on B, is the Area of the Ellipſe on A.

*Examp.* Let it be required to find the Area of an Ellipſe whoſe tranſverſe Axis is 61,6, and conjugate 44,4, then 44.4 × 61,6 × .7854 = 2148.1 is the Area ſought.

Or, *by the Sliding-Rule*, ſet 1,273 on B, to 61.6 on A, then oppoſite 4,44 on B (for 44.4 falls beyond A) is 214.81 on A, and this multiplied by 10, (for Reaſons explained *Part* 1. *Sect.* 4. *Prop.* 3.) gives 2148.1 the ſame as before.

And hence from what was ſaid about the Segment

ment of a Circle, 'twill be eaſy (by *Prop.* 3. or the Table of *Seg.*) to find the Area of any Segment of an Ellipſe, whoſe Abſciſſa (or verſed Sine) is given.

### PROP. VI.

Let $t=2CV$ the tranſverſe Axis of the Hyperbola (*Fig.* 28.) and put $x$ to denote $Vo$, any Abſciſſa thereof whoſe Ordinate $om$ is $y$, then ſhall the Area of $mv$ in that Part whoſe Abſciſſa is $x$, be expreſſed by the following Series where $V=t+x$,

*viz.* $4y \times \frac{x}{1.3} - \frac{x^2}{1.3.5.V} - \frac{x^3}{3.5.7.V^2} - \frac{x}{5.7.9.V^3} - \frac{x^5}{7.9.11.V^4}$ &c.

The Demonſtration of this Series depends on the ſame Principles with that for the Circular Segment *Pag.* 100. and therefore for the ſame Reaſon the Reader muſt take the Inventor's Word for its Truth, till he ſhall think fit to publiſh its Demonſtration.

But we muſt not omit that uſeful Solution given for this Purpoſe by Mr. *Coats*, *Prop.* IV. of his *Harmonia Menſurarum*, and ſince publiſhed by moſt of the Authors that have wrote on Fluxions, particularly my ingenious Friend Mr. *Tho. Simpſon*; tho' I muſt obſerve there is a Preſs-Error in his Solution, as will appear by comparing it with the following one: but if I miſtake not, we were firſt obliged with a Geometrical Demonſtration of the Property of the Hyperbolic Spaces (on which this depends) by that elaborate Geometer *Gregory St. Vincent.* The Theorem is this; If $t$ be the tranſverſe Axis, and $c$ the conjugate of any Hyperbola (*viz.* $c=\sqrt{tp}$) and $x$ any Abſciſſa whoſe Ordinate is $y$, then the Area of the Hyperbola, whoſe Abſciſſa is $x$, will be expreſſed by

$\frac{2+2x}{2}$

$\frac{t+2x}{2} \times y - t \times c \times ,575646 \times L : \frac{t+2x+\frac{2ty}{c}}{t}$ For an Example to both theſe Methods, let $t=30$, $x=10$, $y=12$, then $c = \frac{ty}{\sqrt{tx+xx}} = 18$; hence by the firſt Theorem we have this Proceſs,

$\frac{4xy}{3} = A =$ 160.0000

$\frac{Ax}{5V} = B =$ . 8.0000

$\frac{Bx}{7V} = C =$ ... 2857

$\frac{3Cx}{9V} = D =$ .... 238

$\frac{5Dx}{11V} = E =$ ..... 27

$\frac{7Ex}{13V} = F =$ .... 4

Sum —— 8.3126

The Area 151,6874

And by the ſecond Method, $\frac{t+2x}{2} \times y = 300$, alſo $\frac{t+2x+\frac{cty}{c}}{t} = 3$, and its Logarithm is ,4771212, therefore $t \times c \times ,575646 \times L : 3 = 540 \times ,575646 \times ,4771212 = 148,3126$; this taken from 300, leaves 151,6874, the ſame as was found by the following Series, ſo that the one confirms the other.

Beſides theſe two Methods for meaſuring the Hyperbola, we may add ſome Rules analogous to thoſe for meaſuring the circular Segment: for uſing the ſame Letters as in the Propoſition, the Meaſure of the hyperbolic Segment is nearly expreſs'd by any of theſe three THEOREMS.

I.

I. $\left.\begin{array}{l}\cdot\sqrt{tx}+4\sqrt{tx+\frac{3}{4}x^2}\times\frac{4x}{15}\times\frac{c}{t}\\ 4\sqrt{tx}+21\sqrt{tx+\frac{5x^2}{7}}\times\frac{4x}{75}\times\frac{c}{t}\\ \ldots\ldots\ \frac{80t+39x}{16t+3x}\sqrt{tx}\times\frac{4x}{15}\times\frac{c}{t}\end{array}\right\}=151.648$

II. $=151.681$

III. $=151.602$

The two firſt are taken from the *Analyſis per Quantitatem*, &c. of Sir *Iſaac Newton*, (and publiſh'd by Mr. *Jones*) but the ſecond is neareſt the Truth.

And now we have done with meaſuring Planes, whether contained by Right Lines, or any of the Conic Sections; in the next place we proceed to Solids.

## CHAP. IV.

### *Of Meaſuring Solids.*

#### PROP. I.

*THE Meaſure of every Solid, generated by a Line moving betwixt two parallel equal and ſimilar Planes ſimilarly ſituated (called the Ends thereof) is equal to the Meaſure of one of the Ends, multipled by the Meaſure of a Perpendicular, let fall thence on the other End.*

For let *h* expreſs the Meaſure of that Perpendicular, and *e* the Meaſure of one End or Baſe; then if at the Diſtance of the linear Unit, Planes be drawn thro' the ſolid parallel to its Baſe, it is manifeſt that the ſeveral Elementa contained by every two of theſe contiguous Planes, will be equal to one another; therefore as often as the Baſe contains the *ſuperficial Unit*, ſo often will each Elementa contain

contain the *ſolid Unit*, but $e$ expreſſes that Number. Again, as often as the Perpendicular contains the *linear Unit*, ſo often will the whole Solid contain $e$ ſolid Units; but $h$ is that Number, therefore $h \times e$ is the Number of ſolid Units, or Meaſure of the generated Solid by *Defin.* 23. *Chap.* 1. *Part.* 2. *Q. E. D.*

*Definit.* 1. If the Ends or Planes about which the revolving Line turns be two Squares, each of whoſe Sides is equal to the Diſtance of the Ends, and alſo the generating Line is perpendicular to both of them, the generated Solid is called a *Cube.*

2. If the Ends are Parallelograms, the generated Solid is called a *Parallelopipedon.*

3. If the Ends are Triangles or Polygons of any Number of Sides, the generated Solid is in general called a *Priſm.*

4. If the Ends are Circles, the generated Solid is called a *Cylinder.*

5. If the Ends are any other Curve-Lined Spaces, we may call them *Cylindroids*, prefixing the Name of the Baſe. Thus let them be two Similar, and ſimilarly poſited Ellipſes, then the Solid is an *Elliptical Cylindroid*; if Parabola's, a *Parabolic Cylindroid*; when Hyperbola's, an *Hyperbolic Cylindroid*; all which are meaſured by help of the above Propoſition only, and the aſſiſtance of the Propoſitions foregoing.

*Cor.* 1. For if the generated Solid be a Cube whoſe Side AB (*Fig.* 34.) is $d$, then the Area of the Baſe, or End, is $d^2$, (*Prop.* 1. *Ch.* 1. *Part* 1.) therefore for $e$, $h$, in the above Propoſition, write $d^2$ and $d$, and $e \times h$ becomes $d^3 = d \times d \times d$, whence this Rule for *Meaſuring the Cube.*

Multiply the Meaſure of the Side of the Cube by it ſelf, and that Product again by the Meaſure

of the Side, ſo ſhall this laſt Product be the Meaſure of the Cube ſought.

Or, *by the Lines* D, E *on the Sliding-Rule*, ſet 1 on D to 1 on E, then oppoſite the Side on D is its Meaſure on E.

*Example.* Let the Side of a Cube be 6 Inches,

$$\begin{array}{r} 6 = d \\ 6 \\ \hline 36 = d^2 \\ 6 \\ \hline 216 = d^3 \end{array}$$ the Meaſure ſought.

Or, 1 on D ſet to 1 on E, then againſt 6 on D, is 216 on E, the Content as before; this might be done by the Lines C, D, for one on D ſet to 6 on C, againſt 6 on D, is 216, the Content on C.

*Cor.* 2. Hence if the generated Solid be a Parallelopipedon, whoſe Ends are right-angled Parallelograms, the Meaſures of whoſe Sides are $l$, $b$, and the Meaſure of a Perpendicular let fall from one End to the other (continued if needful) $h$, then $l \times b$, (by *Prop.* 1. *Chap.* I.) is the Meaſure of the Baſe; therefore for $e$ in the Propoſition, write $l \times b$, and we have $l \times b \times h$ for the Meaſure of the Parallelopipedon propoſed.

*Whence this Rule for meaſuring all Parallelopipedons, having rectangular Baſes.* Multiply the Product of the Width and Breadth of the Baſe by the Height of the Solid, and that Product will be its Meaſure.

Or this may be done by the Lines A, B, on the Sliding-Rule at two Operations. Thus 1 on B ſet to the Width on A, againſt the Breadth on B, is the Area or Meaſure of the Baſe on A; then 1 on B ſet to the Area of the Baſe (found above)

on A, againſt the Height of the Solid on B, is its Meaſure on A.

*Examp.* Let there be a Parallelopipedon ABCD, (*Fig.* 35.) whoſe Breath is AB=8 Inches, Width AE=6 Inches, and from ſome Point H in one End, let a Perpendicular be let fall on the other, let it meet the ſame in P, and ſuppoſe HP=12 Inches; then by the above Rule 8×6×12=576 Inches is the Meaſure of the Solid. Or, 1 on B ſet to 6 on A, againſt 8 on B, is 48 on A; againſt which bring 1 on B, then againſt 12 on B is 576 on A, the Meaſure ſought.

*Cor.* Hence the Meaſure of every triangular Priſm, is equal to the Meaſure of a Perpendicular let fall from any Angle of its Baſe on its oppoſite Side, into half that Side, and this Product multiplied by the Height of the Priſm. Thus,

(*Fig.* 36.) Let ABD be a Priſm whoſe Baſe ABC is a Triangle, from an Angle of which as C, let fall a Perpendicular on AB, meeting the ſame in P, alſo let a Plane be drawn thro' any Line *k k* of the End D, perpendicular to the Baſe, and let it meet the ſame in KK, and from any Point H of *kk*, a Line being let fall perpendicular on KK, meeting the ſame in Q, then HQ is the Height of the Priſm: Whence $CP \times \frac{AB}{2} \times HQ$ is the Meaſure thereof, which is evident from this Propoſition, and *Cor.* 2. *Prop.* 1. *Chap.* 1.

*Cor.* 4. If the Baſe was any regular Polygon, let the Area thereof be found by *Corol.* 4. *Prop.* 1. *Chap.* I. then this multiplied by the Height of the Priſm, will be its Content.

*Examp.* Let the Baſe be a Pentagon whoſe Side is 50, the ſame with that *Page* 68. and let the Meaſure of a Line drawn from a Point in the End to the Baſe (*Fig.* 37.) be 12 Inches, then the Area of

of the Baſe is 4301,175, and that multiplied by 12, gives 51614,1 ſquare Inches for the Meaſure of the pentagonal Priſm ABEF. Alſo if the Area of the Baſe was any Conic Section, its Area may be found by one of the three laſt Propoſitions; then this multiplied by the Height, will be the Content of that Priſm.

*Cor.* 5. Hence alſo the Meaſure of every Cylinder is equal to the Square of its Diameter multiplied by ,7854, and that Product by the Height of the Cylinder: But to meaſure this by the Sliding-Rule, another Conſideration muſt be brought; for if the Diameter of the Cylinder's Baſe is $d$, and its Height $h$, then $d^2 \times h, \times ,7854$ is its Content: Call that C, then $d^2 \times h \times ,7854 = C$, or $d^2 \times h = \frac{C}{,7854} = C \times \overline{1.128378}^2$, for $\sqrt{\frac{1}{,7854}} =$ 1.128378, hence $\overline{1.128378}^2 : h :: d^2 : C$. This Proportion gives this following *Rule for meaſuring a Cylinder by the Sliding-Rule.*

Set 1.128378 on D, to the Height of the Cylinder on C, then oppoſite the Diameter of its Baſe on D, is the Content on C. See *Prop.* 5. *Sect.* 4. *Chap.* 3. *Part* 1.

*Examp.* Let the Diameter AB (*Fig.* 38.) of the Cylinder's Baſe be 12 Inches, and its Height 16, to find the Content.

*Firſt*, By the above Solution we have the following Operation.

$d$ = - - - - - - - - - 12
12

$d^2$ = - - - - - - - - 144
0,7854

31416
31416
7854

$d^2 \times ,7854$ = - - 113,0976
16

$d^2 \times h \times ,7854 = C = 1809,5616$

*Or* by the Sliding-Rule, I ſet 1.128 on D, to 16 on C, then oppoſite 1.2 on D is 18,096 on C: and becauſe 12 was divided by 10, I multiply 18,096 by 100, and have 1809.6 for the Meaſure of the Cylinder.

## PROP. II.

*Let* VMN (Fig. 39.) *be a Curve of ſuch a kind, that the Proportion of every Part* VO=x, *and the Perpendicular* MN *(paſſing thro' O, and intercepted by the Curve) being equal to* y, *be expreſſed by the Equation* $y^2 = Ax + Bx^2$ *the Letters* A,B, *denoting given Quantities; then if this Curve revolve round* VO *as an Axis (which from the Equation biſects the Lines* MN, *mn, m'n') the Meaſure of the Solid generated by this Motion will be expreſs'd by this* THEOREM, $\overline{\frac{Ax^2}{2} + \frac{Bx^3}{3}} \times a$,

*(where* a=78539816, *&c.)*

For let the Abſciſſa VO be conceived divided into an infinite Number of equal Parts in the Points *o*, *o'*, *o''*, *o'''*, *o''''*, *&c.* and thro' the ſame imagine Planes, to paſs at right Angles to VO, the

the Axis of Revolution: Then 'tis manifeſt the ſeveral ſolid Elementa $Mn$, $m'n$, $m'n''$, &c. may be taken for ſo many Cylinders having equal Altitudes, the Diameters of their Baſes the Lines MN, $mn$, $m'n'$. $m''n''$, &c. put $q$ to denote the Meaſure of one of the infinitely ſmall Parts $Oo$, $oo'$, $o.o''$, &c, and let us denote the Meaſures of the ſucceſſive elementary Cylinders $Mn$, $mn'$, $m'n''$, &c. by $e$, $e'$, $e''$, $e'''$, &c. reſpectively, and $x$, $x'$, $x''$, $x'''$, &c. for the Diſtance of each them from V, *viz.* $x, x', x'', x'''$, &c. $= Vo$, $Vo'$, $Vo''$, $Vo'''$, &c. hence $\frac{x}{q}$ is the Number of elementary Cylinders, or Abſciſſas $x$, $x'$, $x''$, $x'''$.

Now ſince $y^2 = Ax + Bx^2$, therefore $a \times y^2 = a \times \overline{Ax + Bx^2}$ is the Area of the Baſe of the elementary Cylinder $Mn = e$; therefore by *Cor.* 5. of the laſt Propoſition $q \times a \times y^2 = q \times a \times \overline{Ax + Bx^2} = e$, in which for $x$ write ſucceſſively $x'$, $x''$, $x'''$, &c. and for $e$ write $e'$, $e''$, $e'''$, &c. then we ſhall have the following Table of Elementary Cylinders, and their Values.

*viz.*
$$\begin{cases} e = q \times a \times \overline{Ax + Bx^2} \\ e' = q \times a \times \overline{Ax' + Bx'^2} \\ e'' = q \times a \times \overline{Ax'' + Bx''^2} \\ e''' = q \times a \times \overline{Ax''' + Bx'''^2} \\ \&c. \qquad\qquad \&c. \end{cases}$$

Therefore the Meaſure of the generated Solid is equal to the Sum of a Series of the (1) *one* Powers of Quantit. in Arithm. Progreſſion from $o$ to $Ax$ inclusive, more by the Sum of a Series of the (2) or *ſecond* Powers of Quantit. in Arithm. Progreſſion from $o$ to $Bx^2$, and this Sum multiplied by $q \times a$. Now the Sum of the firſt, *viz.* $Ax + Ax' + Ax'' + Ax'''$, &c. to $o$, is $\frac{Ax}{2} \times \frac{x}{q}$, for by the Lemma *Pag.* 85. the laſt Term is $Ax$, the Number of Terms is $\frac{x}{q}$, and

the Index 1, whence for $n$, $l$, and $m$, write $\frac{x}{q}$, $Ax$ and 2, and we have $\frac{Ax^2}{2q}$ for the Sum required. Alſo the Sum of $Bx^2+Bx'^2+Bx''^2+Bx'''^2$, &c. to $o$, is $\frac{Bx^3}{3q}$ by writing for $n$, $l$, $m$, in the Lemma $\frac{x}{q}$, $Bx^2$, 2; hence then the Meaſure of the whole Solid, which we will call S, is $\overline{\frac{Ax^2}{2q}+\frac{Bx^3}{3q}}\times a\times q$, therefore $\overline{\frac{Ax^2}{2}+\frac{Bx^3}{3}}\times a=$ S. *Q. E. O.*

*Cor.* 1. If in this Expreſſion for $Bx^2$ be put its Value from the Equation of the Curve ($y^2=Ax+Bx^2$) we ſhall find $S=\overline{Ax+2y^2}\times\frac{ax}{6}$, the Meaſure of the Solid generated by the Motion of the Curve VMN about VO.

*Cor.* 2. If any other Abſciſſa as VC be called $b$, and its correſponding Ordinate PQ, $b$, then $\overline{\frac{Ab^2}{2}+\frac{Bb^3}{3}}\times a$ is the Solidity of the Conoid VPQ; whence putting F for the Meaſure of the Fruſtum MNQP, we have $F=3A\times\overline{b^2-x^2}+2B\times\overline{b^3-x^3}\times\frac{a}{6}$; but $b^3-x^3=\overline{b-x}\times\overline{b^2+bx+x^2}$, and $b^2+x^2=\frac{b^2+v^2-A\times\overline{b+x}}{B}$ from the Equation of the Curve, whence

$F=\overline{A\times b+x+2Bbx+2b^2+2y^2}\times\frac{a}{6}\times\overline{b-x}$, which is better accommodated to meaſuring the Fruſtums of Conoids, and conſequently better fitted to Gauging.

From this Propoſition, and the 2 *Cor.* above, we may eaſily deduce the Meaſure of the Cone, Sphere,

Sphere, Spheroids of both kinds, and the two Conoids and their Fruſtums: But firſt we muſt ſhew what is meant by theſe Names.

*Defin.* 1. If the generating Curve VMN (*Fig.* 43.) be a Circle, the generated Solid will be a Sphere, any part of which V*m n* (cut off by a Plane *m n*) from the Vertex, is called the *Segment of a Sphere*; and the difference of this from half of the Sphere, *viz.* MN *m n* is called a *Fruſtum of the Sphere.*

2. If the generating Curve VM*v*N (*Fig.* 42.) be an Ellipſe and the Axe of Revolution the tranſverſe Axis thereof, the generated Solid is called a *Spheroid*; but when the Axe of Revolution is the conjugate Axis, an *oblate Spheroid*, any Part of which V*mn* cut off by a Plane perpendicular to the Axis of Revolution, is called a *Segment* of the Spheroid; and what this wants of half the Spheroid, is called its *Fruſtum*; if the Spheroid is cut by a Plane parallel to the Axis of Revolution, we may call the Piece cut off a *Parallel Segment*; and any Part of the Segment of a Spheroid, cut off by a Plane perpendicular to its Baſe, is called a *ſecond Segment* thereof.

3. If the generating Curve VMN (*Fig.* 41.) be a Parabola, the generated Solid is called a *Parabolic Conoid*; any Part from the Vertex V, (*viz.* V*mn*) cut off by a Plane perpendicular to VO the Axe of Revolution, is called its Segment; and what this wants of the whole Conoid, (*viz. mn*MN) is called its *Fruſtum.*

4. If the generating Curve VMN (*Fig.* 39.) be an Hyperbola, the generated Solid is called an *Hyperbolick Conoid*; and any Part of this from the Vertex, cut off by a Plane perpendicular to the revolving Axe, is called its *Segment*, and what this wants of the whole Conoid, is called an *Hyperbolic Fruſtum.*

5. But if the Circle or any of the Conic Sec-

tions revolve round an Ordinate to the Axis of the Circle, or Conic Section, the Solid in general is called a *Spindle*, with the Name of the generating Curve prefix'd. Thus if the generating Curve be a

| | | | |
|---|---|---|---|
| *Circle*, *Ellipſe*, *Parabola*, *Hyperbola*, | The generated Solid is a | *Circular* *Elliptic* *Parabolic* *Hyperbolic* | Spindle. |

*Cor.* 5. From the preceeding Propoſitions it follows, if the Height of any Cone (*viz.* VO *Fig.* 40.) be $b$, the Diameter of its Baſe (*viz.* MN) $= b$, then the Meaſure thereof ſhall be $ab^2 \times \frac{b}{3}$. And if this Cone be cut thro' $o$, any Point in the Axis by a Plane parallel to its Baſe, ſo that the Meaſure of V$o$ be $x$, and that of the Line $mn$, $y$, then the Meaſure of the Fruſtum $mn$MN is $\overline{b^2+by+y^2} \times a \times \frac{b-x}{3} = \frac{b^3-y^3}{b-y} \times \frac{a \times \overline{b-x}}{3}$.

For let the Cone be generated by the right-angled Triangle VON, round VO as an Axis, then *per* ſimilar Triangles $x : y :: b : b$, therefore $by = bx$, or $y^2 = \frac{b^2x^2}{b^2}$, this being compared with the Equation in the Propoſition (*viz.* $y^2 = Ax + Bx^2$) gives $A = o$, $B = \frac{b^2}{b^2}$; therefore $S = \overline{Ax+2y^2} \times \frac{ax}{6}$ (*Corollary* 1.) is $S = \frac{y^2 \times ax}{3}$; this when $x$ becomes $b$, and $y = b$, is $S = \frac{b^2 \times a \times b}{3}$; the Meaſure of the Cone AMN. Alſo $F = \overline{A \times \overline{b+x} + 2Bbx + 2b^2 + 2y^2} \times \overline{b-x} \times \frac{a}{6}$ becomes $F = \overline{\frac{2b^2}{b^2} \times bx + 2b^2 + 2y^2} \times \frac{\overline{b-x} \times a}{6}$, or $F = \overline{b^2+by+y^2} \times \frac{\overline{b-x} \times a}{3}$ ( for $\frac{2b^2x}{b}$

$\frac{2b^2x}{b} = 2by$, from the Equation of the Cone) and from hence the following Rules for meaſuring any Cone, and its Fruſtum univerſally having a circular Baſe, both by the Pen and Sliding-Rule.

Rule 1. *To meaſure any Cone, having a circular Baſe.*

The Square of the Cone's Baſe being multiply'd by its Height, and that Product by ,26179939, or in Practice by 2618, the Reſult will be the Meaſure ſought.

Rule 2. *To meaſure any Cone by the Sliding-Rule.*

Set 1.9544 on D to the Cone's Height on C, then oppoſite the Diameter of the Baſe on D is the Content of the Cone on C.

For an Example of theſe two Rules: Let the Hieght of a Cone be 50 Inches, and the Diameter of its Baſe 30.

*The Operation.*

| | |
|---|---|
| $b =$ - - 30 | ,2618 |
| 30 | $45000 = b^2 \times b$ |
| $b^2 =$ 900 | 13090000 |
| $b =$ - 50 | 10472 |
| $b^2 \times b = 45000$ | $11781,0000 = b^2 \times b \times ,2618$ |

Or, by the Sliding-Rule 1.9544 on D being ſet to 50 on C, then againſt 3 on D (for 30 is of the Line) is 117.81; which multiplied by 106, the Square of the Diviſor of 30 (ſee *Prop.* 5. *Sect.* 2. *Chap.* I.) gives 11781, for the Meaſure of the Cone as before.

Rule 3. *To meaſure the Fruſtum of a Cone by the Pen.*

To the greater Diameter of the Fruſtum and the Leſſer; and from the Square of this Sum, take the Product of the two Diameters; multiply the Re-

Remainder by the Height of the Fruſtum, and that Product by ,2618 ; ſo ſhall this laſt Product give the Meaſure of the Fruſtum ſought.

Rule 4. *To meaſure the Fruſtum of a Cone by the Sliding-Rule.*

Set 2,764 on D to the Height of the Fruſtum on C, then find both the greater and leſſer Diameters, and alſo their Difference on D, and note the three Numbers oppoſite them on C: then from triple the Sum of the firſt and ſecond, take the third, and the Remainder is the Meaſure ſought.—The Reaſon of this Rule is from hence, that $b^2 + by + y^2 = \frac{3b^2 + 3y^2 - \overline{b-y}^2}{2}$.

*Example* of a Fruſtum of the above Cone. Let the Height of the Fruſtum be 20 Inches, then the Diameter is 18 Inches.

*The Operation.*

| | |
|---|---|
| 30 | $=b$ |
| 18 | $=y$ |
| 48 | $=b+y$ |
| 48 | |
| 384 | |
| 192 | |
| 2304 | $=\overline{b+y}^2$ |
| 540 | $=b \times y$ |
| 1764 | $=\overline{b+y}^2 - b \times y$ |
| 20 | $=b$ |
| 35280,0 | $=\overline{\overline{b+y}^2 - b \times y} \times b$ |
| 8162 | $=$ ,2618 inverted |
| 70560 | |
| 21168 | |
| 353 | |
| 282 | |
| Meaſ. of Conic Fru. 9236,3 | $=\overline{\overline{b+y}^2 - b \times y} \times b \times$ ,2618 |

Or

Or, by the Sliding-Rule 2,764 on D being ſet to 20 upon C, againſt 30,18 and 12 upon D is 2356.2; 848,23 and 377 reſpectively: the Sum of the firſt and ſecond is 3204,43, three times which is 9613,3; from which the third being taken, leaves 9236.3 as before.

*Cor.* 4. Alſo from this Propoſition it follows, if the Height of any parabolic Conoid be VO=$b$, (*Fig.* 41.) the Diameter of its Baſe $b$, and in the Axis any Point $o$ be taken, diſtant from V by $x$, and a Plane be drawn thro' the ſame, whoſe Section with the Conoid is the Circle $m\,n$, the Diameter of which is $m\,n=y$, then the Meaſure of the Conoid VMN will be $\frac{ab^2\times b}{2}$; and the Meaſure of the Fruſtum $mn$MN $=\overline{b^2+y^2}\times a\times\frac{b-x}{2}$. For (*per Cor.* 1. *Prop.* 2. *Chap.* 2.) $b:b^2::x:y^2$, therefore $y^2=\frac{b^2}{b}\times x$, this compared with the Equation ($y^2=Ax+Bx^2$) in the Propoſition gives $A=\frac{b^2}{b}$, $B=o$, whence $S=A\overline{x+2y^2}\times\frac{ax}{6}=\overline{\frac{b^2}{b}\times x+2y^2}\times\frac{ax}{6}$, but $\frac{b^2}{b}=y^2$, hence $S=3y^2\times\frac{ax}{6}=\frac{y^2\times ax}{2}$, and this when $x$ becomes $b$ is $\frac{b^2\times a\times b}{2}=$VMN: alſo $F=\overline{A\times\overline{b+x}+2Bbx+\overline{2b^2+2y^2}\times a\times\frac{b-x}{6}}$ becomes $\overline{\frac{b^2}{b}\times\overline{b+x}+2b^2+2y2}\times a\times\frac{b-x}{6}$, or $F=\frac{\overline{b^2+b^2x+2b^2+2y^2}}{b}=\frac{\overline{3b^2+3y^2}\times a\times\overline{b-x}}{6}=\overline{b^2+y^2}\times a\times\frac{b-x}{2}=mn$MN.

And from hence we have the following Rules for meaſuring the parabolic Conoid and its Fruſtum.

Rule

Rule 1. *To meaſure a parabolic Conoid by the Pen.*

Let the Square of the Conoid's Baſe be multiplied by its Height, and this Product by ,3927, the laſt Product is the Meaſure of the Conoid propoſed.

Rule 2. *To meaſure a parabolic Conoid by the Sliding-Rule.*

Set 1.59577 on D, to the Conoid's height on C, then againſt the Diameter of its Baſe on D, will be the Meaſure ſought on C.

Rule 3. *To meaſure the Fruſtum of a parabolic Conoid by the Pen.*

To the Square of the greater Diameter add the Square of the Leſſer, and multiply their Sum by the Height; this Product multiply'd by .3927, is the Meaſure ſought.

Rule 4. *To meaſure the Fruſtum of a parabolic Conoid by the Sliding-Rule.*

Set 1.59577 on D to the Conoid's Height on C, and find the greater and leſſer Diameters on D, the Sum of the Numbers oppoſite them on C, is the Meaſure of the Fruſtum required.

*For an Example to theſe four Rules*; there is a parabolic Conoid, whoſe Height is 60 Inches, the Diameter of the Baſe 40 Inches, and which is cut at the Diſtance of 21.6 Inches from its Vertex by a plane Parallel to its Baſe, to find the Content both of the Conoid VMN, and its Fruſtum *mn*MN*mn*: it will be found that $mn=y=24$ for $60 : 40 \times 40 :: 21.6 : 24 \times 24 = \overline{mn}^2$, thence $mn = 24$, and $oO = 38.4$ the Height of the Fruſtum.

By the *firſt Rule* $s = \frac{a \times b^2 \times h}{2}$.

And by the *third Rule* $F = \overline{b^2 + y^2} \times a \times \frac{h - x}{2}$.

The

The OPERATION.

For the whole Conoid.

$40 = b$

$40$

$1600 = b^2$

$60 = b$

$96000,0 = b^2 \times b$

$7293 = 3927$ Inverted.

$288000$

$86400$

$1920$

$672$

$37699,2 = b^2 b \times, 3927$

$y = 24$

$24$

$96$

$48$

$y^2 = 576$

For the Fruſtum.

$40 = b$

$40$

$1600 = b^2$

$576 = y^2$

$2176 = b^2 + y^2$

$38,4 = b$

$8704$

$17408$

$6528$

$835 8,4 = \overline{b^2 + b^2 \times b}$

$7293 =, 3927$ Inverted.

$250675$

$75203$

$1670$

$585$

$328 13,3 = \overline{b^2 + b^2} \times b \times, 3927$

For the whole Conoid by the Sliding-Rule thus, by *Rule* 2. I ſet 1.59577 on D, to 40 on C, then againſt 6 on D (for 60 is off the Rule) is 376.99 on C; this multiply'd by 100, the Square of the Diviſor of 60, gives 37699, the Meaſure of the Fruſtum as before.

To meaſure the Fruſtum of the Conoid by the Sliding-Rule, thus by *Rule* 4. I ſet 1.59577 on D to 38.4

38.4 on C; then againſt 2.4 and 4 on D, is 86.86=$\alpha$ and 241.27=$\beta$, the Sum of which is $\alpha+\beta$=328.13, which I multiply by 100 the Square of the Diviſors of 24 and 40, and then find 32813, for the Meaſure of the Conoid, as found before.

*Cor.* 5. If the generating Curve be the Semi-Ellipſis VMN (*Fig.* 42.) about its tranſverſe $Vv=2b$, and the Conjugate $MN=b$, and if this be cut by a Plane, parallel to the Baſe MON, which is diſtant from V by $Vo=x$, whoſe Section with MVN, is the Line $mn=y$, then the Meaſure of the

*Segment*, $Vnnm=\overline{\frac{b^2x}{b}+y^2}\times\frac{ax}{3}=\overline{3y^2+\frac{b^2x^2}{b^2}}\times\frac{ax}{6}$

*Semi-Spheroid* $VMNM=2b^2\times\frac{ab}{3}$

*Fruſtum mn* $MN=\overline{2b^2+y^2}\times\frac{\overline{b-x}\times a}{3}$

For (by *Prop.* 1. *Chap.* 2. *Part.* 1.) $b^2:b^2::\overline{2b-x}\times x:y^2=\frac{2b^2x}{b}-\frac{b^2x^2}{b^2}$: this Compared with $y^2=Ax+Bx^2$ (the Equation of the Propoſition) gives $A=\frac{2b^2}{b}$, $B=-\frac{b^2}{b^2}$, and therefore by *Cor.* 1. $S=\overline{\frac{b^2x}{b}+y^2}\times\frac{ax}{3}=Vmnm$ the Segm. In which making $x=b$ and $y=b$, we have $2b^2\times\frac{ab}{3}=VMNM$ the Semi-ſpheroid; and by (*Cor.* 2.) we have $F=\overline{\frac{2b^2}{b}\times\overline{b+x}-\frac{b^2x}{b}+2b^2+2y^2}\times\frac{\overline{b-x}\times a}{6}=2b^2+y^2\times\frac{\overline{b-x}\times a}{3}$, the Fruſtum *mn*MN, and from hence the three following Rules for meaſuring theſe Solids by the Pen.

*Rule*

Rule 1. *To meaſure any Segment of a Spheroid.*

Let the Square of the conjugate Axis of the Spheroid be multiplied by the Height of the Segment, and call the Quotient thereof divided by the Semi-tranſverſe $a$, to which add the Square of the Diameter of the Segment's Baſe; and multiply the Sum by the Height of the Segment, and that Product by ,2618, which will give the Meaſure of the Segment propoſed.

Rule 2. *To meaſure any Spheroid.*

Multiply the Square of the conjugate Axis of the generating Ellipſe by the tranſverſe Axis thereof, and that Product by ,5236, ſo ſhall you have the Meaſure of the Spheroid propoſed.

Rule 3. *To meaſure any Fruſtum of a Spheroid.*

To twice the Square of the Diameter of the greater End, add the Square of the Diameter of the leſſer End; let this Sum be multiply'd by the Height or Length of the Fruſtum, and that Product by ,2618, which will give the Meaſure of the Fruſtum ſought.

*Examp.* Let the tranſverſe Axis be $40=2b$, the Conjugate $30=b$, the Height of the Segment $8=x$, then $b^2 : b^2 :: \overline{2b-x} \times x : y^2 = 576$, and $b-x=12$.

OPERATION.

*For the Segment.*

$b =$ - - - - - 30
30

$b^2 =$ - - - 900
$x =$ - - - - - 8

$b =$ - - 20) 7200 (360 $= \frac{b^2x}{b}$

$\frac{b^2x}{b} =$ - - - 360
$y^2 =$ - - - 576

$\frac{b^2x}{b} + y^2 =$ - 936
8

$\frac{b^2x}{b} x + y^2 \times x =$ 7488,0000
,2618 inverted 21620

14976000
4492800
74880
59904

$mVn =$ - - 1960.3584

Fol

OPERATION *for the Spheroid.*

| | |
|---|---|
| $b$ = | 30 |
| | 30 |
| $b^2$ = | 900 |
| $2b$ = | 40 |
| $b^2 \times 2b$ = | 36000 |
| | ,5236 |
| | 216000 |
| | 108 |
| | 72 |
| | 180 |
| $b^2 \times 2b \times .5236$ = VMN = | 18849,6000 |
| From its half = | 9424,8000 |
| Take the Segment = | 1960,3584 |
| The Remainder = | 7464,4416 |

is the Fruſtum as found by the next Operation.

OPE

OPERATION *for the Fruſtum.*

| | |
|---|---|
| $b =$ - - - - | 30 |
| | 30 |
| $b^2 =$ - - - | 900 |
| | 2 |
| $2b^2 =$ - - - | 1800 |
| $y^2 =$ - - - | 576 |
| $2b+y^2 =$ - - | 2376 |
| $b-x =$ - - | 12 |
| | 4752 |
| | 2376 |
| $2b^2+y^2 \times b-x =$ - | 28512.000 |
| 2618 inverted - | 8162 |
| | 57024000 |
| | 17107200 |
| | 285120 |
| | 228096 |
| $mn\,MN =$ - - | 7464,4416 |

And from hence we have the three following Rules by the Sliding-Rule.

Rule 1. *To meaſure the Segment of a Spheroid by the Sliding-Rule.*

Firſt ſet the Semitranſverſe on B to the Conjugate on A, and call the Number $\alpha$, which is on A oppoſite the Height of the Segment on B; then ſet

ſet 2.764 on D to the Height of the Segment on C, and find both the Number $\alpha$, and the Diameter of the Segment's Baſe on D, then if to the Number on C oppoſite $\alpha$ be added triple the Number oppoſite the ſaid Diameter, that Sum will be the Meaſure of the Segment; ſo in the preſent Caſe, firſt I ſet 20 on B to thirty on A, then oppoſite 8 on B is 12 on A, which is $\alpha$: then ſetting 2,764 on D to 8 on *c* againſt 1,2 (for 12 is off the Rule) is 1,508, and againſt 2,4 is 6,0318, which triple, is 18,0954; this added to 1,508 gives 19.6034, which multiplied by 100 (the Square of the Diviſor of $\alpha$ and 24) gives 1960.34, nearly the ſame as was found before.

Rule 1. *To meaſure a Spheroid by the Sliding-Rule.*

Set 1.382 on D to the tranſverſe Axis on C, then againſt the conjugate Axis on D is the Meaſure of the Spheroid on C.

Thus in the preſent caſe 1.382 on D being ſet to 40 on C againſt 3 on D (for 30 is off the Rule) is 188.5 on C; this multiplied by 100 (the Square of the Diviſor of 30) gives 18850. for the Meaſure of the Spheroid, nearly the ſame as before.

Rule 3. *To meaſure the Fruſtum of a Spheroid by the Sliding Rule.*

Set 1.954 on D to the Height or Length of the Fruſtum on C, then the greater and leſſer Diameters of the Fruſtum's Ends being found D, to twice the Number oppoſite the firſt on C, add that oppoſite the leſſer on C, and this Sum will be the Meaſure of the Fruſtum ſought.

So here, 1,954 on D being ſet to 12 on C, againſt 3 on D is 28.274 on C, and againſt 2.4 on D is 18.096 on C, to which add 56.548 the double

of 28,274, and you have 74.644; where removing the Decimal Point two Places towards the right Hand (becauſe both 30 and 24 were divided by 10) we find 7464.4 for the Meaſure required; nearly the ſame as before.

*Cor.* 6. If in the Expreſſion above for the Spheroid, its Segment and Fruſtum, you put $b = \frac{b}{2}$; you'll have the following Expreſſions for the Sphere, its Segment and Fruſtum, (*vide Fig.* 43.) where MN=$b$. the Meaſure of the

Segment V*mnn* is $\overline{2bx+y^2} \times \frac{ax}{3} = \overline{3y^2+4x^2} \times \frac{ax}{6}$.

Semiſphere VMNM is $\frac{b^3 \times a}{3}$.

Fruſtum *mn*MN is $\overline{2b^2+y^2} \times \overline{b-2x} \frac{xa}{6}$.

And theſe Expreſſions put into Words, give the three following Rules for meaſuring the Sphere, its Segment and Fruſtum by the Pen.

Rule 1. *To meaſure the Segment of a Sphere.*

To the Square of the Diameter of the Segment's Baſe, add twice the Product of the Diameter of the Sphere by the Height of the Segment, and multiply this Sum by the Height of the Segment; then this laſt Product multiplied by ,2618 gives the Meaſure of the Segment propoſed.

Rule 2. *To Meaſure any Sphere.*

The Cube of the Sphere's Diameter multiplied by ,5236, gives the Meaſure thereof.

Rule 3. *To meaſure the Fruſtum of a Sphere.*

To twice the Square of the Sphere's Diameter, add the Square of the end; and multiply the Sum by

by the Height of the Fruſtum, and that Product by ,2618; ſo will you have the Meaſure ſought.

*Examp.* Let the Diameter of a Sphere be 25 Inches, the Height of a Segment thereof 9 Inches, and conſequently the Diameter of its Baſe 24 Inches, and the Height of the Fruſtum 3.5 Inches.

OPERATION *for the Segment.*

| | |
|---|---|
| $y =$ | 24 |
| | 24 |
| | 96 |
| | 48 |
| $y^2 =$ | 576 |
| $2bx = 50 \times 9 =$ | 450 |
| $2bx + y^2 =$ | 1026 |
| $x =$ | 9 |
| $\overline{2bx + y^2} \times x =$ | 9234,0000 |
| 2618 inverted | 8162 |
| | 18468000 |
| | 5540400 |
| | 92340 |
| | 73872 |
| $mVn =$ | 2417,4612 |

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OPERATION *for the Sphere.*

| | |
|---|---|
| $b =$ - - | 25 |
| | 25 |
| | 125 |
| | 50 |
| $b^2 =$ - - - | 625 |
| | 25 |
| | 3125 |
| | 1250 |
| $b^3 =$ - - - | 15625,0000 |
| .5236 inverted | 6325 |
| | 78125000 |
| | 3125000 |
| | 468750 |
| | 93750 |
| $b^3 \times ,5236$ | $=$8181.2500 |

OPE-

OPERATION *for the Fruſtum.*

| | |
|---|---|
| $b =$ | 25 |
| | 25 |
| | 125 |
| | 50 |
| $b^2 =$ | 625 |
| | 2 |
| $2b^2 =$ | 1250 |
| $y^2 =$ | 576 |
| $2b^2+y^2 =$ | 1826 |
| $\frac{b-2x}{2} =$ | 3.5 |
| | 9130 |
| | 5478 |
| $\sqrt{2b^2+y^2} \times \sqrt{b-2x} =$ | 6391,000 |
| .2618 inverted | 8162 |
| | 1278 2000 |
| | 383 4600 |
| | 63910 |
| | 51128 |
| $\sqrt{2b^2+y^2} \times \sqrt{\frac{b-2x}{2}} \times ,2618 =$ | 1673,1638 |

And from theſe Rules we may deduce three o-thers, for meaſuring theſe Solids by the Sliding-Rule.

Rule 1. *To meaſure a ſpherical Segment by the Sliding-Rule.*

Set 2.764 on D to the Height of the Fruſtum on C, and find the Diameter of the Segment Baſe, and alſo the Height of the Segment on D; then to triple the Number on C oppoſite the firſt, add quadruple that oppoſite the ſecond, ſo ſhall this Sum be the Meaſure required.

So in the above *Example*, 2.764 on D ſet to 9 on C, againſt 2.4 on D, is 6,7866 on C, and againſt 2.5 is 95.43; now becauſe 2.5 was divided by 10, 5678,6 is the firſt Number, which tripled is 2035,8; and 95.43 quadrupled is 381.72, theſe added together, give 2417.52, the Meaſure ſought, nearly the ſame as before.

Rule 2. *To meaſure a Sphere by the Sliding-Rule.*

Set 2.382 on D to the Diameter of the Sphere on C, then againſt the ſaid Diameter on D, is the Meaſure ſought on C.

In the preceding *Example*, I ſet 1,382 on D to 2,5 on C; then againſt 2.5 on D is 81,812, in which removing the decimal Point two Places forwards (becauſe 2.5 is $\frac{1}{10}$ of 25) and we have 8181.2 for the Meaſure of the Sphere.

Rule 3, *To meaſure the Fruſtum of a Sphere by the Slyding-Rule.*

Set 1.954 on D to the Height of the Fruſtum on C, and find the Meaſures of the Diameters of the Ends on D, then twice the Number on C againſt the greater, added to that againſt the leſs, is the Meaſure required.

In the preſent Caſe 1,954 on D, ſet to 3.5 on C; then againſt 25 on D is 572.69, which doubled is

1145.38

1145,38, and againſt 24 is 527,79, which added to 1145,38 gives 1673,17 nearly the ſame as before.

*Cor.* 7. If the generating Curve be the Hyperbola PVQ (*Fig.* 39.) about the Axis VC=$b$, the Diameter PQ=$b$, and ſuppoſe this cut by a Plane parallel to the Baſe PQ, diſtant from the Vertex V by VO=$x$, the double Ordinate MN=$y$, and the tranſverſe Axis of the Hyperbola equal to $t$; then the Meaſure of the Segment VMN $= \frac{3t+2x}{t+b} \times \frac{abyz}{2}$, Fruſtum PQNM $= \frac{3t+2b}{t+b} \times bb^2 - \frac{3t+2x}{t+x} \times xy \times \frac{a}{6}$. For call the Parameter of the Hyperbola $p$, then (by *Prop.* 3. *Chap.* 2.) $t : p :: tx+x^2 : \frac{y^2}{4}$; therefore $y^2 = 4px + \frac{4px^2}{t}$. Let this Equation be compared with that in the Propoſition, then $A = 4p$, and therefore (by *Cor.* 1.) $S = \sqrt{4px+2y^2} \times \frac{ax}{6}$; and this when $x$ becomes $b$, and conſequently $y=b$ is PVQ $= \sqrt{4pb+2b^2} \times \frac{ab}{6} = \frac{3t+2b}{t+b} \times \frac{abb^2}{6}$; for the ſame reaſon MVN $= \frac{3t+2x}{t+x} \times \frac{axy^2}{6}$; and the Difference of theſe is the Meaſure of the Fruſtum PQMN $= \sqrt{\frac{3t+2b}{t+b} - xy^2} \times \frac{3t+2x}{t+x} \times \frac{a}{6}$. From hence it is manifeſt to meaſure the hyperbolic Conoid, beſides the Data that was required in the preceding ones, we muſt know the Meaſure of the tranſverſe Axis of the generating Hyperbola; and to meaſure the Fruſtum, the whole Height of the Conoid muſt be found from the Equation of the Curve; that is the *Sphere*, *Spheroid* and *parabolic*

*lic Conoid* are in a determinate Ratio to their *circumſcribing Cylinders*, but in the hyberbolic Conoid, that Ratio is variable.

We ſhall add no Example to this, becauſe it ſeldom can occur in Gauging; and from what has been ſaid above, it will be eaſy for the Reader to ſupply that of himſelf.

*Cor.* 8. If the generating Curve be a Semi-Ellipſe VN*v* (*Fig.* 42.) about its *Semiconjugate* Axis NO, which now let be *b*, and its Tranſverſe *b*, thereby generating the *Semioblate Spheroid* VN*v*, which is cut by a Plane *n n'* parallel to V*v*, at the Point *o*, in the revolving Axis, diſtance from N the Vertex, by N*o*=*x*; and the correſponding Ordinate to this, *viz.* *nn'* be put equal to *y*, then the Meaſure of the Segment *n*N*n'* is

$$\overline{\frac{b^2x}{b} + y^2} \times \frac{ax}{3} = \overline{\frac{b^2x^2}{b^2} + 3y^2} \times \frac{ax}{6};$$

The Semioblate Spheroid VN*v* is $\frac{2b^2 \times ab}{3}$;

The Fruſt. *n n'v*V is $\overline{2b^2 + y^2} \times \overline{b - x} \times \frac{a}{3}$.

Which are analogous to thoſe for the common Spheroid, its Segment, and Fruſtum *Cor.* 5. foregoing; for here the Equation of the generating Curve is $y^2 = \frac{2b^2x}{b} - \frac{b^2}{b^2}x^2$; whence for A = $\frac{2b^2}{b}$, and B = $\frac{b^2}{b^2}$, (*Cor.* 1 and 2, foregoing) theſe Values being put, you'll have the Expreſſions above.

Having in the preceding Propoſition, and its Corollaries, given the Meaſure of the Cone, Conoid, their Fruſtums and Segments, we ſhall now ſhew how to find the Meaſure of a *ſecond Segment*, or Slice cut off by a Plane, parallel to the Axe of Revolution; in order to which, we muſt premiſe the following Lemmas.

LEMMA

## LEMMA I.

*If a Parabolic Conoid be cut by a Plane, parallel to the Axis of Revolution, the Figure of the Section will alſo be a Parabola, and the Parameter thereof ſame with that of the Generating Parabola.*

For let MVN (*Fig.* 44.) be the Conoid, and *mvn* the Section thereof, by a Plane parallel to VO the Axis of Revolution, which cuts the generating Parabola, in the Line *v o*; then will *mnv* be a Parabola, whoſe Parameter is equal to that of MVN; call it *p*, then (by *Cor.* 3. *Prop.* 2. *Chap.* 2.) $p \times vo = Mo \times oN$, but $Mo \times oN = \overline{on}^2$, (8 *Euclid* 6.) therefore $p \times vo = \overline{on}^2$, which is the Property of a Parabola, whoſe Parameter is *p*, Abſciſſa *v o*, and ordinate *on*; therefore, *&c.*

## LEMMA II.

*If a Spheroid be cut by a Plane, parallel to the Tranſverſe or Axis of Revolution, the Figure of the Section will alſo be an Ellipſis, the Tranſverſe and Conjugate of which will be to thoſe of the generating Ellipſe in a direct Proportion.*

For let VM*v*RM (*Fig.* 46.) be the Semiſpheroid, generated by the Semi-Ellipſe VR*v* about its Tranſverſe Axis *Vv*, whoſe Center is Q, and ED its Section by a Plane parallel thereto, ſo that E*mn*D*mn* is the Section of that Plane and the Spheroid: Let two Planes be drawn perpendicular to the Tranſverſe, one of which paſſes thro' the Center Q, making the Semicircle MRM, and the other thro' any Point *q* (in *Vv*), making the Circle N*r*N, and cutting the Plane E*mn*D*mn*E, in the Lines *mm*, *nn*; I ſay E*mn*D*nm*E is an Ellipſis, and ED its Tranſverſe, is to *mm* its Conjugate, as *Vv* is to 2OR.

For

For put VQ=T, QR=C, OD=*t*, O*m*=*c*, O*o*=*x*=*on*=*y*; then from the Property of the Circle $\overline{rq+qo} \times \overline{rq-qo} = \overline{rq}^2 - \overline{oq}^2 = \overline{on}^2 = y^2$; but (by *Cor.* 2. *Prop.* 1. *Chap.* 2.) $\overline{rq}^2 = C^2 - \frac{C^2x^2}{T^2}$, and for the ſame Reaſon $\overline{oq}^2 = \overline{EF}^2 = C^2 - \frac{C^2t^2}{T^2}$; therefore $y^2 = \frac{C^2}{T^2} \times \overline{t^2 - x^2}$, where making $x = o$ then $y = c$, but then $c^2 = y^2 = \frac{C^2t^2}{T^2}$, or $c = \frac{Ct}{T}$, that is T : C :: *t* : *c*, whence $y^2 = \frac{c^2}{t^2} \times \overline{t^2 - x^2}$, which (by *Cor.* 2. *Propoſit.* 1. *Chap.* 2.) is the Equation of an Ellipſe, whoſe Tranſverſe is 2*t*, and Conjugate 2 *c*.

## Prop. III.

*If the Parabolic Conoid* MVN (Fig. 44.) *whoſe Axis is* VO=h *the Diameter of its Baſe* b, *be cut by a Plane* mvn *parallel to* V⊙, *and diſtant from it by* oO=z, *the Meaſure of the Slice* mnvM *ſhall be expreſſed by this*

## Theorem, *viz.*

$mnv\,M = \frac{S}{2} - \frac{z \times b^2 - 4z^2 \sqrt{b^2 - 4z^2}}{6b^2} \times h$, where S denotes the Meaſure of the circular Segment *m*M*n*, *viz.* of a Circle, whoſe Diameter is *b*, and verſed Sine $\frac{b-2z}{2}$.

For put $mn = y = \sqrt{b^2 - 4z^2}$, then becauſe by the firſt Lemma above, *mvn* is a Parabola, whoſe Para-

Parameter is $\frac{b^2}{4b}$, we have (by *Prop.* 2. *ch.* 2. *part* 2.) $\frac{b^2}{4b} \times v\,o = \frac{y^2}{4}$: but $\frac{2vo \times mn}{3}$ is the Meaſure of *mvn*, (by *Prop.* 4. *Chap.* 3.) therefore $\frac{2by^2}{b^2} \times \frac{y}{3} = \frac{2by^3}{3b^2}$ is equal to *mvn*.

Let this be drawn into $\frac{y\,\dot{y}}{2\sqrt{b^2-y^2}}$ the Fluxion of $Mv = \frac{b-\sqrt{b^2-y^2}}{2}$, and we have $\frac{b}{3b^2} \times \frac{y^4\dot{y}}{\sqrt{b^2-y^2}}$ for the Fluxion of the Meaſure ſought; but $\frac{y^2\dot{y}}{2\sqrt{b-^2y^2}}$ $=$S. Now from the 7th *Propoſition* of the Book of Quadratures, if P be the Area of a Curve, whoſe Ordinate is $py^{\theta-1} \times \overline{e+fx^n}^{\lambda-1}$, and Q the Area of another Curve, whoſe Ordinate is $q^{\theta+n-1} \times \overline{e+fy^n}^{\lambda-1}$, then $Q = \frac{q}{p} \times \frac{py^{\theta} \times \overline{e+fy^n}^{\lambda} - e\theta P}{f \times \lambda n+\theta}$: therefore becauſe here $p = \frac{1}{2}$, $q = \frac{b}{rb^2}$, $\theta = 3$, $\lambda = \frac{3}{2}$, $e = b^2$, $f = -1$, and $n = 2$, we have $Q = \frac{P}{2} - y^3\frac{\sqrt{b^2-y^2}}{12b^2} \times b$, in which for P write S, and for $y$, $b^2-z^2$, then $\frac{S}{2} - \frac{z \times b^2-4z^2\sqrt{b^2-4z^2} \times b}{6b^2}$ is the Meaſure of the Slice *mnv*M. *Q. E. O.*

*Cor.* 1. Let M*n*N*m* (*Fig.* 45.) be equal to the Baſe of the Conoid, MN its Diameter, and M*o* $=$M*o*; thro' *o* draw the Perpendicular *mon*, meeting the Circle in the Points *m*, *n*, from *m*. Thro' the Center O draw the Line *m*O*k*, and from *n*, let fall a Perpendicular thereon, meeting the ſame

in

in $l$, then the Solidity of the Slice $mnv$M of the Conoid (*Fig.* 44.) may be expreſt by $\overline{\frac{mn\mathrm{M}}{2} - \frac{mnl}{6}} \times b$, for (*per Figure*) $b : y :: y : \frac{y^2}{b} = ml$, and $b : \sqrt{b^2 - y^2} :: y : \frac{y\sqrt{b^2-y^2}}{b} = nl$; therefore $\frac{y^2}{2b} \times \frac{y\sqrt{b^2-y^2}}{b} = y^3 \frac{\sqrt{b^2-y^2}}{2b^2}$ is the Meaſure of the Triangle $mnl$; conſequently $\overline{\frac{mn\mathrm{M}}{2} - \frac{mnl}{6}} \times b$ is equal to $\overline{\frac{\mathrm{P}}{2} - \frac{y^3\sqrt{b^2-y^2}}{12b^2}} \times b = mnv\mathrm{M}$.

*Cor.* 2. If $nl$ (*Fig.* 45.) be continued till it meet the Circumference in $w$, and the Line $mw$ drawn, then the Segment $n\mathrm{M}mw = 2mn\mathrm{M} + 2nml$, whence $\overline{8mn\mathrm{M} - n\mathrm{M}mv}$ $(= \overline{6mn\mathrm{M} - 2mnl} \times \frac{b}{2} = \overline{\frac{mn\mathrm{M}}{2} - \frac{mnl}{6}} \times b)$ is the Meaſure of $mnv\mathrm{M}$, which is the Theorem publiſh'd by Mr. *Hunt*, and ſaid to be received from Sir *Iſaac Newton*; but what we have given above is more ſimple, it requiring but the Area of one Segment of a Circle to be found, whereas by this Method you muſt have the Area of two.

*For an Example*, Let MN = 92 Inches, M$o$ = 31 Inches, and V$o$ = 294 Inches, then by the Theorem (*Pag.* 100.) I find the Meaſure of the Segment $m\mathrm{M}n$ equal to 1968 nearly, and becauſe M$o$ = 31, $z$ = 15, therefore $\frac{z \times b^2 - 4z^2 \times \sqrt{b^2-4z^2}}{6b^2} = \frac{15 \times 7564 \times \sqrt{7564}}{6 \times 84164} = 194.31$ *proxime*; hence according to the Theorem $\frac{1968}{2} - 19431 = 789.69 \times 294 =$

294=232169 nearly, the Meaſure of the Slice *mn*V*m*. *Q. I. E.*

In order to ſhew the Invention of the above Theorem, I was obliged to uſe the Method of Fluxions, becauſe beſides the *Lemma*, *Pag.* 85, I muſt have inſerted ſeveral more (too tedious for this Place) in order to render a Demonſtration but tolerably clear.

## PROP. IV.

*If a Spheroid generated from the Ellipſe* VQvR, *whoſe tranſverſe Axis is* Vv=t (Fig. 46.) *and Semiconjugate* $QR=\frac{c}{2}$ *be cut by a Plane thro'* O *in* QR, *diſtant from* R, *the Vertex of the Conjugate by* RO=*x*, *and* ED *the Section of the generating Ellipſe be called* y, *the Meaſure of the Segment of the Spheroid* EDR *ſhall be expreſſed by this* THEOREM, *viz.* $EDR=\overline{2t^2x+cy^2}\times\frac{ax}{3t}$.

For conceive OR divided into an Infinite Number of Parts, the firſt Diviſion of which from O is at O′, and let *q* denote the Meaſure of the ſmall Part OO′, now 'tis manifeſt (by *Lemma* 2. foregoing) E*m*D*m*E, the Section of the Cutting-Plane with the Spheroid, is an Ellipſe ſimilar to the generating Ellipſe VR*v*; therefore $t:c::y:\frac{cy}{t}=mn$ the Conjugate of the Section E*m*D*m*E, but becauſe MRM (ſee *Lemma* 2.) is a Circle whoſe Diameter is *c*, we have $mn=2\sqrt{cx-x^2}$, therefore $\frac{cy}{t}=2\sqrt{cx-x^2}$ : but all ſimilar Figures are as the Square of their homologous Sides, (as is demonſtrated further on) and the Area of the generating

rating Ellipſe is $axtxc$, (by *Cor.* 1. *Pr.* 5. *Ch.* 3.) therefore $c^2 : axtxc :: 4x\overline{cx-xx} : \frac{4at}{c} x \overline{cx-x^2}$ the Meaſure of the Ellipſe E*m*D*m*E, and this drawn into $p$, gives the Meaſure of the firſt or greateſt Elementa of the Segment ſought, call it $e$, and the ſucceeding ones $e'$, $e''$, $e'''$, &c. the correſponding Values of RO, $x'$, $x''$, $x'''$, &c. Then we have the Meaſure of the ſeveral Elementa, as here under.

$$e = \frac{4qat}{c} \times \overline{cx-x^2}$$

$$e' = \frac{4qat}{c} \times \overline{cx'-x'^2}$$

$$e'' = \frac{4qat}{c} \times \overline{cx''-x''^2}$$

$$e''' = \frac{4qat}{c} \times \overline{cx'''-x'''^2}$$

and the Sum of theſe is the Meaſure of the whole Segment ſought, but the Number of the Elementa is $\frac{x}{q}$, and the Quantities $x$, $x'$, $x''$, $x'''$, &c. are in arithmetic Progreſſion; hence (by the *Lemma* *P.* 85.) $\frac{4qat}{c} \times \frac{cx^2}{2q} - \frac{4qat}{c} \times \frac{x^3}{3q} = \frac{4at}{c} \times \overline{\frac{cx^2}{2} - \frac{x^3}{3}}$, is the Meaſure of $RDmEm = \frac{2atx}{3c} \times \overline{3cx - 2x^2}$: in which for $x$ put its Value $cx - \frac{c^2y^2}{4t^2}$, and we have $\overline{2t^2x+cy^2} \times \frac{dx}{3t} = RDmEm$. *Q. E. O.*

*Corol.* 1. Therefore making $x = \frac{c}{2}$ and $y = t$, then for the Meaſure of half the Spheroid we have $\overline{2t^2x}$

$2t^2 \times \frac{c}{2} + ct^2 \times \frac{ac}{6t} = \frac{a \times tc^2}{3}$, the ſame as was ſhewn before, *Cor.* 5. *Prop.* 2.

*Cor.* 2. If from the Meaſure of the Semiſpheroid $\frac{a \times tc^2}{3}$, that of the Segment, *viz.* $\overline{2t^2 x + cy^2} \times \frac{ax}{3t}$, be taken, we ſhall have for the Meaſure of the Fruſtum this

THEOREM, *viz.* $VvDE = \overline{2t^2 + y^2} \times \frac{a \times c}{3t} \times \frac{c - 2x}{2}$.

And the firſt of theſe Theorems being turn'd into Words, give the following *Rule for meaſuring a parallel Segment of a Spheroid.*

RULE.

Multiply the Square of the Tranſverſe Axis of the generating Ellipſe by twice the Height of the Segment; and the Square of the Tranſverſe of the Section of the Spheroid, by the Conjugate of the generating Ellipſe, and add theſe two Products together, then multiply their Sum by the Product of ,2618 by the Height of the Segment; this laſt Product divided by the tranſverſe Axis of the generating Ellipſe, is the Meaſure of the Segment ſought.

*Examp.* Let the Tranſverſe of the generating Ellipſe be 73 Inches, its Conjugate 29 Inches, the Height of the Segment 2 Inches, and therefore the Tranſverſe of the Section 25 Inches, to find the Meaſure of the Segment RD*m*E*m*.

OPERATION.

| | | | |
|---|---|---|---|
| $y=$ | - 37 | 73 | $=t$ |
| | 37 | 73 | |
| | 259 | 219 | |
| | 111 | 511 | |
| $y^2=$ | - 1369 | 5329 | $=t^2$ |
| $c=$ | - 29 | 4 | $=2x$ |
| | 12321 | 21316 | $=2t^2x$ |
| | 2738 | | |
| $cy^2=$ | 39701 | 39701 | $=cy^2$ |
| | | 61017 | $=2t^2x+cy^2$ |
| | | 2 | $=x$ |
| $3t=219)$ | | 122034 | $557,20=\overline{2t^2x+cy^2}\times\frac{x}{3t}$ |
| | | 1095 | 4587,0 |
| | | 1253 | 39004 |
| | | 1095 | 4458 |
| | | | 279 |
| | | 1584 | 22 |
| | | 1533 | |
| | | 51 | $436,63=\overline{2t^2x+cy^2}\times\frac{ax}{3t}$ |

*Corol.* 3. But if the above Segment ſhould be cut by a Plane through ſome Point $r$, and parallel to the Conjugate RO, then the Piece cut off is called a *Second Segment* of the Spheroid; to meaſure which by a finite Expreſſion, is impoſſible, even tho' you uſe the Segment of a Circle; at leaſt

leaſt by any Method I yet have met with: but when *o*D is ſmall in reſpect to OD, it may be had by an Approximation, ſufficiently near for Practice, from what has been ſaid in the laſt Propoſition: for in the Figure put $rq = \frac{b}{2}$, $qo = z$, $oD = l$, and denote the circular Segment *nrnon* by S, then the Meaſure of the ſecond Segment D*rnn* is nearly had by this

THEOREM, *viz.* $Drnn = \overline{\frac{3b^2 \times S}{b^2 - 4z^2} - z\sqrt{b^2 - 4z^2}} \times \frac{l}{6}$.

*Corol.* 4. From hence, and *Prop.* 3. we may compute the Meaſure of the Part *m*R*mnnr* (*Fig.* 46.) for by changing the Form of that Theorem, *viz.* for $x$ put its equal $\frac{c - 2z}{2}$, from this Suppoſition of $QO = qo = z$, then $\frac{t}{c} = \frac{2L}{\sqrt{c^2 - b^2}}$, and $l = \frac{t}{2c}\sqrt{c^2 - 4z^2} - \frac{L}{2}$: ſo that the Meaſure of the ſecond Segment O*o*R*r*, which is nearly $\overline{c + z \times c - 2z}^2 \times \frac{ta}{rc} + z\sqrt{b^2 - 4z^2} - \frac{3b^2 \times S}{b^2 - 4z^2} \times \frac{l}{6}$, may be computed from the Meaſures of RQ, *rq*, O*o*, and OQ ($= oq$) being given; that is the Bung and Head-Diameters of a Spheroidal Caſk being given, and alſo its Length, and the Diſtance of the Surface of the Liquor from the Axis thereof, we may by this *Corol.* nearly compute how much is drawn off: But Note, if the Caſk be more than half empty, you will then find how much remains in it; which taken from the whole Content, leaves what is drawn off. But this Theorem requires too

much labour to be followed by the Officers of Exciſe, ſo we ſhall give other Methods for that purpoſe further on.

But I muſt obſerve this is exceeding near the Truth, eſpecially if for S you put $\frac{121b+78z}{29b+6z} \times \frac{\overline{b-2z}}{15} \times \sqrt{2b \times b-2z}$, which is deduced from that *Page* 101. by putting $\frac{b-2z}{2}$ for *v*.

We might alſo give a like Approximation for the *Second Segments* of the Hyperbolic Conoids, for they as well as thoſe of the Spheroid, do not admit of a perfect Cubation; but as no Caſks are ever made reſembling this Solid, I ſhall not detain the Reader with uſeleſs Speculations, but proceed to ſhew how to meaſure the Spindles, which are generated by the Revolution of a Conic Section round an Ordinate to the Axis.

## Prop. V.

*Let* AVBvA (Fig. 47.) *be a Spindle, generated by the Parabola* AVB*'s turning round its double Ordinate* AB; *and thro'* n *a Point in the Axe of Revolution, let a Plane be imagined to paſs perpendicular to it, whoſe Section with the Spindle is the Circle* Mnm; *(put* Vv=b, Mm=y *and* Cn=x) *then the Meaſure of the Fruſtum* VMmv *ſhall be expreſſed by this*

### Theorem, *viz.*

$$VMmv = 2b^2 + y^2 - \tfrac{2}{5} \cdot \overline{b-y}^{\frac{1}{2}} \times \frac{ax}{3}.$$

For call the *Parameter* of the generating Parabola *p*, and ſuppoſe C*n* divided into an infinite Number of Parts, each of which is meaſured by *q*; and denote the ſucceſſive Values of *x* from *n*

to

to C by $x'$, $x''$, $x'''$, &c. *viz.* $x'=x-q$, $x''=x'-q$, $x'''=x''-q$, &c. alſo ſet $e$, $e'$, $e''$, $e'''$, &c. denote the correſponding Elementa of the Solid.

Now (by *Prop.* 2. *Chap.* 2.) $p \times \overline{VO}=\overline{MO}^2$, that is $p \times \frac{b-y}{2} = x^2$, therefore $y=b-\frac{2x^2}{p}$, and $y^2=b^2-\frac{4bx^2}{p}+\frac{4x^4}{p^2}$; whence the Value or Meaſure of of the firſt Elementa ($e$) is $aq \times \sqrt{b^2-\frac{4bx^2}{p}+\frac{4x^4}{p^2}}$; and from hence the following Equations;

$$e = aq \times \sqrt{b^2-\frac{4bx^2}{p}+\frac{4x^4}{p^2}}$$

$$e' = aq \times \sqrt{b^2-\frac{4bx'^2}{p}+\frac{4x'^4}{p^2}}$$

$$e'' = aq \times \sqrt{b^2-\frac{4bx''^2}{p}+\frac{4x''^4}{p^2}}$$

$$e''' = aq \times \sqrt{b^2-\frac{4bx'''^2}{p}+\frac{4x'''^4}{p^2}}$$

&c. &c.

And ſince $q$ *Diſtance* gives 1 Term, $x$ *Diſtance* will give $\frac{x}{q}$ *Number of Terms*; that is $\frac{x}{q}$ is the Number of Elementa, or Terms of the above Series, each of which is alſo compoſed of three Terms; the Sum of all the *firſt* is $aq \times b^2 \times \frac{x}{q} = ab^2 \times x$, and (by the *Lemma*, *Pag.* 85.) the Sum of all the *ſecond* Terms, *viz.* $aq \times \frac{4b}{p} \times \sqrt{x^2+x'^2+x''^2}$, &c. is $aq \times \frac{4b}{p} \times \frac{x^2}{3} \times \frac{x}{q} = \sqrt{\frac{4abx^3}{3p}}$,

and (by the ſame *Lemma*) the Sum of all the *third* Terms, *viz.* $aq \times \frac{4}{p^2} \times \overline{x^4 + x'^4 + x''^4}$, &c. is $aq \times \frac{4}{p^2} \times \frac{x^4}{5} \times \frac{x}{q} = \frac{4ax^5}{5p^2}$, whence collecting theſe Sums under their reſpective Signs, we have $VMmv = \overline{b^2 - \frac{4bx^3}{3p} + \frac{4x^5}{5p^2}} \times ax$: but $\frac{2x^2}{p} = b - y$, therefore the laſt Theorem is equivalent to $\overline{b^2 - \frac{2b \times \overline{b-y}^2}{3} + \frac{\overline{b-y}^4}{5}} \times ax$, that is to $\frac{8b^2 + 4by + 3y^2}{5} \times \frac{a}{3}$, and this to $\frac{10b^2 + 5y^2 - 2b^2 - 2y^2 + 4by}{5} \times \frac{ax}{3} = \overline{2b^2 + y^2 - \frac{2}{5} \cdot \overline{b-y}^2} \times \frac{ax}{3}$. *Q. E. O.*

*Cor.* 1. Whence if $y = o$, then the Meaſure of the whole Spindle is $\overline{2b^2 - \frac{2}{5}b^2} \times \frac{a \times AB}{3}$, or $\frac{8}{15} \times b^2 a \times AB$, that is $\frac{8}{15}$ths of its circumſcribing Cylinder.

From this Propoſition we have the following Rules.

Rule 1. *To Meaſure the Fruſtum of a parabolic Spindle.*

To twice the Square of the Diameter of the greater End, add that of the leſſer, and from the Sum take $\frac{2}{5}$ths, or (,4 =) $\frac{4}{10}$ths of the Square of their Difference, and multiply the Remainder by the Height of the Fruſtum; then if this Product be multiplied by ,2617994 = 2618 nearly, it will give the Meaſure ſought.

Rule 2. *By the Sliding-Rule.*

Set 1,954 on D, to the Height of the Fruſtum on C; then find the greater and leſſer Diameters, as

as alſo their Difference on D, noting the three Numbers oppoſite them on C; then if the ſecond is added to twice the firſt, and $\frac{2}{5}$ths, or $\frac{4}{10}$ths of the third taken from the Sum, you'll have the Meaſure requir'd.

*Examp.* Let (*b*) the greater Diameter be 32 Inches, (*y*) the leſſer 24, and (*x*) the Length 40 Inches.

OPERATION.

$32 = b$
$32$
$64$
$96$
$1024 = b^2$
$2048 = 2b^2$
$y = 24$
$24$
$96$
$48$
$576 = 576 = y^2$
$2624 = 2b^2 + y^2$
$,4 \times 8.8 = 25,6 = 25,6 = \frac{2}{5} \cdot \overline{b - y}^2$
$2598,4 = 2b^2 + y^2 - \frac{2}{5} \cdot \overline{b - y}^2$
$40 = x$
$103936,0$
$8162,0 = ,2618$ inverted.
$207872$
$62361$
$1039$
$831$
$27210,4 = \overline{2b^2 + y^2 - \frac{2}{5} \cdot \overline{b - y}} \times \frac{ax}{3}$

But by taking $a=2617994$ inſtead of ,2618, we ſhould find the Meaſure to be 27210,3814 true to the laſt Figure.

*By the Sliding-Rule.* When 1.954 on D is ſet to 40 on C, then againſt 32, 24 and 8 on D, is 10723,3, 6031,9 and 670,2 reſpectively; therefore,

| | | |
|---|---|---|
| Twice the Firſt - - | = | 21446,6 |
| More by the Second - | = | 6031,9 |
| | | 27478,5 |
| Leſs by $\frac{4}{10}$ths the Third | = | 268,08 |
| Gives the Content - | = | 27210,42 |

## PROP. VI.

*Let* AVBvA (Fig. 48.) *be a Spindle generated by the elliptic Space* AVB *round* AB, *which is parallel to* TT, *the tranſverſe Axis thereof; and thro' a Point in* AB, *diſtant from* C, *the Middle of the Spindle by* x, *let a Plane be drawn perpendicular thereto, whoſe Section, with the generated Solid, is the Circle* Mnm: *and put* d=OC, *the Diſtance of* AB, *from* TT, *and* S *for the Meaſure of the elliptic Space* MV*o*; *then the Meaſure of* VMmvV, *a Fruſtum of the Solid, ſhall be expreſſed by this*

## THEOREM, *viz.*

$$\text{VM}mv\text{V}=2b^2+y^2+8d\times b-y-\frac{3S}{x}\times\frac{ax}{3}.$$

For put $t$ for the Tranſverſe, and $c$ the conjugate Axis of the generating Ellipſis, the reſt as in the preceding Propoſition, then (by *Prop.* 1. *Chap.* 2.) $t^2 : c^2 :: t^2-4x^2 : \overline{2d+y}^2=\overline{Mm}^2$, therefore $y=\frac{c}{t}\sqrt{t^2-4x^2}-2d$, and conſequently $y^2\times q\times a$, the

the Meaſure of the firſt Elementa is, $qax\sqrt{c^2+4d^2-\frac{4c^2x^2}{t^2}}-\frac{4dc}{t}\sqrt{t^2-4x^2}$; and from hence the following Equations.

$$e=qax\sqrt{c^2+4d^2-\frac{4c^2x^2}{t^2}}-\frac{4dc}{t}\sqrt{t^2-4x^2}$$

$$e'=qax\sqrt{c^2+4d^2-\frac{4c^2x'^2}{t^2}}-\frac{4dc}{t}\sqrt{t^2-4x'^2}$$

$$e''=qax\sqrt{c^2+4d^2-\frac{4c^2x''^2}{t^2}}-\frac{4dc}{t}\sqrt{t^2-4x''^2}$$

$$e'''=qax\sqrt{c^2+4d^2-\frac{4c^2x'''2}{t^2}}-\frac{4dc}{t}\sqrt{t^2-4x'''^2}$$

*&c.* *&c.*

Now (by the *Lemma*, *Pag.* 85.) 'tis manifeſt, the Sum of all the three firſt Terms of theſe Elementa, is $ax\,c^2x+4d^2x-\frac{4c^2x^3}{3t^2}$, and the Sum of the fourth Terms are evidently equal to $8da\times$ MVON.

But the Meaſure of MVON is $S+x\times d+\frac{y}{2}$, which being put for MVON, and $c^2-\overline{2a+y}^2$ for its Equal $\frac{4c^2x^2}{t^2}$, then we ſhall have the Meaſure of $VMmv=2c^2+\overline{2d+y}^2-12d^2-12dy\times\frac{ax}{3}-8dxS$; but $d+\frac{b}{2}=\frac{c}{2}$, therefore $2c^2=8d^2+8db+2b^2$; put this for $2c^2$ in the laſt Expreſſion, then for the Meaſure of VM*mv* we ſhall find this

THEOREM,

THEOREM, *viz.*

$VMmv = 2b^2 + y^2 + 8d \times b - y - \frac{3S}{x} \times \frac{ax}{3}$. *Q.E.O.*

*Cor.* 1. If $t = c$, or the Ellipſe in this Propoſition ſhould be chang'd into a Circle, then the elliptic Space VM*o* becomes the Circular Segment VM*o*, (*Fig.* 49.) and therefore for *Meaſuring of the Fruſtum of a circular Spindle*, we have this

THEOREM.

$VMmv$ (*Fig.* 49.) $= 2b^2 + y^2 + 8d \times b - y - \frac{3S}{x} \times \frac{ax}{3}$.

*Cor.* 2. If (in the Theorem for meaſuring an Elliptick Spindle) for $d$ be wrote $-d$, becauſe O, *Fig.* 48. is on the contrary ſide C, to what it is *Fig.* 51. *Then for meaſuring the Fruſtum of an Hyperbolic Spindle we have this*

THEOREM.

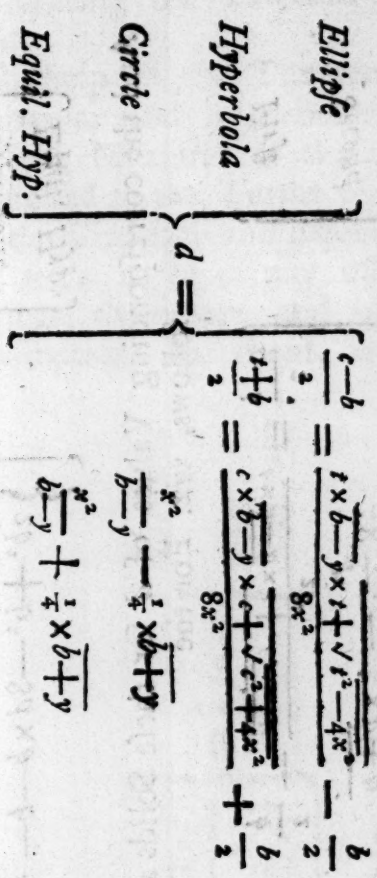
$VMmv$ (*Fig.* 50.) $= 2b^2 + y^2 - 8dxb - y - \frac{3S}{x} \times \frac{ax}{3}$,

where $b = Vv$, $y = Mm$, $x = Cn$, $d = CO$ and S the Meaſure of the hyperbolic Space VM*o*.

But if any are not ſatisfied with theſe Inferences, they may by the ſame Steps with thoſe for the Elliptic Spindle, inveſtigate the two laſt Theorems *à priori*.

*Cor.* 3. If $t$ be the tranſverſe, and $c$ the conjugate Axis of the Ellipſe or Hyperbola, then for the

*Ellipſe*




But if $l$ be put to denote the Line $nn$, or $l=2x$, that is, if $l$ denote the Length of a Caſk, generated by a conic Section, round an Ordinate to the Axis, whoſe Bung-Diameter is $b$, Head $b$, and $Q$ denote that Part of the generating Curve, whoſe Abſciſſa is $\frac{b-b}{2}$, and Baſe or double Ordinate $l$, then the Meaſure of theſe Caſks will be found by the following Theorems, *viz.*

The

$$\text{The } \left\{ \begin{array}{l} \text{Elliptic} \\ \text{Circular} \\ \text{Hyperbolic} \\ \text{Equil. Hyp.} \end{array} \right\} \text{Spindle} \left\{ \begin{array}{l} = \sqrt{2b^2+b^2+8d\times b-b-\frac{3Q}{l}\times\frac{al}{3}} \\ = \sqrt{2b^2+b^2-8d\times b-b-\frac{3Q}{l}\times\frac{al}{3}} \end{array} \right.$$

And the corresponding Values of $d$ for these Solids are as follows, *viz.* For the

$$\left. \begin{array}{l} \text{First} \\ \text{Second} \\ \text{Third} \\ \text{Fourth} \end{array} \right\} d = \left\{ \begin{array}{l} \frac{c-b}{2} = \frac{t\times b-b\times t+\sqrt{t^2-l^2}}{2l^2}\times\frac{b}{2} \\ \frac{t+b}{2} = \frac{t\times b-b\times t+\sqrt{t^2+l^2}}{2l^2}\times\frac{b}{2} \\ \frac{l^2}{\frac{1}{4}\times b-b}-\frac{1}{4}\times b+b \\ \frac{l^2}{\frac{1}{4}\times b-b}+\frac{1}{4}\times b+b \end{array} \right.$$

*Cor.* 4. By making $d=0$, we find the same Theorems for the Fruſtums of a Sphere and Spheroid, that was given at *Cor.* 5, 6. of the ſecond Propoſition.

And hence it alſo follows, if the Hyperbola MVM (*Fig.* 50.) revolve round NN, a Line paſſing thro' the Center of the Hyperbola, the Meaſure of the generated Solid MMM′M′ is $2\times\overline{V v'}^2+\overline{MM'}^2\times\frac{a\times NN}{3}$, which is an Expreſſion ſimilar to

thoſe

thoſe for meaſuring the Fruſtum of the Sphere and Spheroid.

*Cor.* 5. By help of the Theorems for *Approximating* the circular and hyperbolic Segments, we might give the Meaſures of theſe Solids without Q, *viz.* expreſſed in the Terms of the Queſtion: but as only the Circular, and Equilateral hyperbolic Spindles, ſeem to be of any uſe in Gauging, we have omitted the others, and only put down the Approximations for them, which are thus, *viz.* the

$$\left.\begin{matrix}\text{Circular}\\ \text{Equil. Hyp.}\end{matrix}\right\}\ \text{Spindle} = \left\{\begin{matrix}2b^2+b^2-\dfrac{14\times\overline{b-b}^2\times\overline{l^2+b^2-b^2}}{35l^2+5\times\overline{b-b}^2}\\ \hline 2b^2+b^2-\dfrac{14\times\overline{b-b}^2\times\overline{l^2+b^2-b^2}}{35l^2-5\times\overline{b-b}^2}\end{matrix}\right\}\times\frac{al}{3}\ \text{nearly.}$$

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For

*For an Example to these two Cases*; Let $b=32$, $h=24$, and $l=40$ Inches, then for the

$$\left.\begin{matrix}\textit{Circular}\\ \textit{Eq.Hyp.}\end{matrix}\right\} \textit{Spind.}\ d = \left\{\begin{matrix}\frac{1}{4}\times\frac{1600}{8}-\frac{1}{4}\times 56\\ \frac{1}{4}\times\frac{1600}{8}+\frac{1}{4}\times 56\end{matrix}\right\} = \left\{\begin{matrix}50-14=36\\ 50+14=64\end{matrix}\right.$$

Twice the first added to $b$, or 32, gives 104 for the Diameter of the generating Circle; and $b$, or 32, taken from twice the second, leaves 96 for the transverse Axis of the generating Hyperbola; whence from the Theorems, *Prop.* 3, and 6, of *Chap.* 2. For the

$$\left.\begin{matrix}\textit{Circle}\\ \\ \textit{Hyperb.}\end{matrix}\right\} Q = \left\{\begin{matrix}\frac{320}{3}+\frac{1}{5}\times\frac{1}{25}\,A-\frac{1}{7}\times\frac{1}{25}\,B+\frac{3}{9}\times\frac{1}{25}\\ \quad C-\frac{5}{11}\times\frac{1}{25}\,D+\ \&c.\\ \frac{320}{3}-\frac{1}{5}\times\frac{1}{25}\,A-\frac{1}{7}\times\frac{1}{25}\,B-\frac{3}{9}\times\frac{1}{25}\\ \quad C-\frac{5}{11}\times\frac{1}{25}\,D-\ \&c.\end{matrix}\right.$$

that is, $\frac{320}{3} = A = -\ 106.666667$

$$\begin{array}{rl}
\frac{1}{5}\times\frac{1}{25}\,A = B = & -\ 853333\\
\frac{1}{7}\times\frac{1}{25}\,B = C = & -\ 4876\\
\frac{3}{9}\times\frac{1}{25}\,C = D = & -\ -\ 65\\
\frac{5}{11}\times\frac{1}{25}\,D = E = & -\ 1\\
\hline
B+C+D+E = & -,858275
\end{array}$$

$A-B-C-D,\ \&c. = 105,808392 = MVM$

*Fig.* 50. the Area of the Hyperbolic Segment.

| | |
|---|---|
| $A = 106,666667$ | |
| $B = 853333$ | $,004876 = C$ |
| $D = 65$ | $1 = E$ |
| $A+B+D = 107,520065$ | $,004877 = C+D$ |
| $004877$ | |
| $107,515188$ | $= A+B-C+D-E$ |

The Area of the Circular Segment MVM *Fig.* 49.

Whence

Whence by the above Theorems, we have for the *Circular Spindle* this

OPERATION.

$32=b$
$32$
$64$
$96$
$1024=b^2$
$2$
$2048=2b^2$
$576=y^2$ $=576$
$2624=2b^2+y^2$

$24=y$
$24$
$96$
$48$
$576$

$107,51518=Q$
$3$
$l=40)\ 322,54554=3Q$
$(8,06364=\frac{3Q}{l}$
$8,00000=b-y$
$-,06364=b-y-\frac{3Q}{l}$
$288=8d$
$50912$
$50912$
$12728$
$-18,32832=8dx\overline{b-y-\frac{3Q}{l}}=18,32832$

$2605,67168=2b^2+y^2+8dx\overline{b-y-\frac{3Q}{l}}$
$40$
$104226,8672$
$883997162,0$ $(=a)$ *inverted.*

$208453734$
$62536120$
$1042269$
$729588$
$93804$
$9380$
$313$
$83$
$8$

$27286,5299=\overline{2b^2+y^2+8dx\overline{b-b-\frac{3Q}{l}}}\times\frac{al}{3}$

And

And for the equilateral *Hyperbolic Spindle* this

OPERATION.

105,80839 = Q

3

$l$=40) 317,42517 = 3Q

(7,93563 = $\frac{3Q}{l}$

8,00000 = $b-h$

0,06437 = $b-h-\frac{3Q}{l}$

512 = $8d$

12874
6437
32185

32,95744 = $8d \times \overline{b-h-\frac{3Q}{l}}$

2624 . . . = $2b^2+h^2$

2591,04256

40 = $l$

103641,70240 = $\overline{2b^2+h^2-8d \times \overline{b-h-\frac{3Q}{l}}} \times l$

883997162,0 = 261799, &c. inverted.

2072834048
621850214
10364170
7254919
932775
93277
3109
829
83

27133,33424 = $\overline{2b^2+h^2-8d \times \overline{b-h-\frac{3Q}{l}}} \times \frac{al}{3}$

I muſt obſerve the Value of Q in the Rule for meaſuring the Hyperbolic Spindle, might have been computed from Mr. *Coats*'s Theorem for that purpoſe, *Page* 110; but the Series in this Caſe does it with leſs trouble.

*Or*, By the two Theorems for meaſuring theſe Solids by Approximation, we have for the

1ſt. $\sqrt{2624 - \frac{14.64.1152}{56320}} \times ,2618 \times 40 = 27286,53$

2d. $\sqrt{2624 - \frac{14.64.2048}{55680}} \times ,2618 \times 40 = 27133,35$

Nearly the ſame as before.

## SCHOLIUM I.

We ſhall now collect together the ſeveral Theorems found by *Propoſit.* 2, 5, and 6; which include all the Varieties of Caſks, hitherto mentioned by thoſe who have wrote on Gauging, and ſet down the Theorems, and their Contents for Mr. *Everards's* Caſk, whoſe Bung-Diameter is $b=32$, Head $h=24$, and Length $l=40$ Inches: The Names of which are, *1ſt*, The Middle Fruſtums of a *Spheroid*; *2d*, *Circular Spindle*; *3d*, *Hyperbolic Spindle*; *4th*, *Parabolic Spindle*; 5th, Two Fruſtums of a *Parabolic Conoid*; and *6th*, two Fruſtums of a *Cone*.

| | | | |
|---|---|---|---|
| 1 | $2b^2+b^2 \times \frac{al}{3}$ | | |
| 2 | $2b^2+b^2+8d \times b-b-\frac{3Q}{l} \times \frac{al}{3}$ | | |
| 3 | $2b^2+b^2-8d \times b-b-\frac{3Q}{l} \times \frac{al}{3}$ | = | 27133.334=96.2175 |
| 4 | $2b^2+b^2-\frac{2}{5} \times \overline{b-b}^2 \times \frac{al}{3}$ | | 27210,381=96.4907 |
| 5 | $b^2+b^2 \times \frac{al}{2}$ | | 25132,741=89.1232 |
| 6 | $b^2+b \times b \times b \times \frac{al}{3}$ | | 24797,638=87.9349 |

*Cub. Inch.* *Ale Gall.*

27478,464=97.4414

27286,530=96.7607

The Meaſures here differ a ſmall matter from what Mr. *Everard* gave in his *Appendix*, which ariſes from his inductionary Method of computing their Contents, but the true Theorems for Meaſuring the circular, elliptic and hyperbolic *Spindles* were never, that I know of, publiſh'd before, if we except the laſt, for which a Theorem is given by a Gentleman (that writes himſelf *Merones*) in the *Mathematical Diary* for 1740.

I muſt obſerve the Hyperbola in the above Example was ſuppoſed to be Equilateral, becauſe otherwiſe the *Data* was not ſufficient, but the Theorem

orem itſelf is univerſal, for all Hyperbolic Spindles whatever.

### Scholium II.

To what has been already ſaid for meaſuring Caſks, we may add the Method for finding what is call'd the *mean Diameter*; for tho' Rules for that purpoſe are to be met with in moſt Books on Gauging, yet I cannot find any that have ſhewn whence they were derived, nor is this Defect all that makes it neceſſary here to examine into this Matter, but chiefly to prevent the Errors that muſt unavoidably ariſe from uſing thoſe fixt Numbers call'd *Multipliers:* for there neither are, nor is it poſſible there ſhould be ſuch, that will reduce all Caſks of the ſame Name and Length, to Cylinders, but they ought to vary as the Proportion betwixt the Bung-Diameter, and the Difference of this from the Head varies. 'Tis true indeed that Proportion may be ſuppoſed pretty near the ſame in moſt Caſks to be met with in Practice, which is all that many are ſolicitous about; this I allow, however, ſince they are not abſolutely ſo, 'tis neceſſary, nay I am perſuaded every Officer of the leaſt Ambition, to have a *rational* Knowledge of the Art he profeſſes, muſt be deſirous to know when the fixt Multipliers may be depended on, and when not, and alſo whence they were obtained.

That ready *Analyſt*, Mr. *Tho. Simpſon*, in his Treatiſe of Fluxions, has given the *Limits* betwixt which are the Numbers we are ſpeaking of, and from what he has done, with little Trouble the very Multipliers themſelves may be found: However, as his Book may not be in the Hands of every one concern'd in Gauging, and alſo ſome of them may want the neceſſary Knowledge to

make ſuch Inferences, I have not only given the *very Rules themſelves* for finding the Multipliers, but alſo the *Limits* from a different Conſideration, which I ſuppoſe will be underſtood with leſs Trouble, as I have no occaſion for the Term *Infinite*, which that Gentleman is oblig'd to introduce in determining one of them.

In order to which, let $b$ be the Bung-Diameter of any Caſk, in the Shape of the Fruſtum of a *Spheroid*, *Parabolic Spindle*, *Parabolic Conoid*, or *Cone*, $h$ the Head Diameter, and $d=b-h$: Alſo ſuppoſe $m$ ſuch a Number or Quantity, that multiplying $b-h$, or $d$, and the Product *taken from* $b$, the Bung, the Remainder may be the Diameter of a Cylinder, equal in Length to the Caſk propos'd and of equal Magnitude; and put $m'$ to denote the Multiplier when the Product is to be *added* to $h$, the Head Diameter, in order to have the *Mean* ſought. Then 'tis evident $m'=1-m$, for ſince $\overline{b-m\times b-h}=\overline{h+m'\times b-h}$, we have $\overline{b-h}=\overline{m+m'\times b-h}$, or $m'=1-m$.

But to find $m$ (by the preceding Scholium and *Prop.* 1. *Chap.* 4.) we have theſe Equations.

*viz.* For two Fruſtrums of a

| | |
|---|---|
| *Spheroid* | $\frac{2b^2+h^2}{3}$ |
| *Parab. Spin.* | $\frac{2b^2+h^2-\frac{2}{5}d^2}{3}$ |
| *Parab. Con.* | $\frac{b^2+h^2}{2}$ |
| *Cone* | $\frac{b^2+h^2+bh}{3}$ |

$\}=\overline{b-md}^2$

In which for $h$ put its value $b-d$, and for $\frac{b}{b-h}$ write $r$; then from the Equations above, we ſhall reſpectively have the following ones, *viz.*

$\frac{1}{r}=$

$$\frac{1}{r} = \left\{\begin{array}{l} \frac{m^2-2}{3m^2-1} \\ \frac{30m-10}{15m^2-3} \\ \frac{4m^2-2}{2m^2-1} \\ \frac{6m-3}{3m^2-1} \end{array}\right\} \text{ and } m = \left\{\begin{array}{l} r-\sqrt{r^2+\frac{1-2r}{3}} = \frac{1}{3}-\frac{1}{3}\times\frac{b-b}{2b+b} \\ r-\sqrt{r^2+\frac{3-10r}{15}} = \frac{1}{3}-\frac{4}{15}\times\frac{b-b}{5b+b} \\ r-\sqrt{r^2+\frac{1-2r}{2}} = \frac{1}{2}-\frac{1}{4}\times\frac{b-b}{b+b} \\ r-\sqrt{r^2+\frac{1-3r}{3}} = \frac{1}{2}-\frac{1}{12}\times\frac{b-b}{b+b} \end{array}\right\} \textit{proxime.}$$

And these Values of $m$ taken from *Unity*, leave those of $m'$ as hereunder,

Two Fruſtums of a

| | | |
|---|---|---|
| *Spheroid* | $m' =$ | $\frac{2}{3}+\frac{1}{3}\times\frac{b-b}{2b+b}$ |
| *Parabol. Spindle* | | $\frac{2}{3}+\frac{4}{15}\times\frac{b-b}{5b+b}$ |
| *Parab. Con.* | | $\frac{1}{2}+\frac{1}{4}\times\frac{b-b}{b+b}$ |
| *Cone* | | $\frac{1}{2}+\frac{1}{12}\times\frac{b-b}{b+b}$ |

*nearly.*

From hence it appears there can be no fixt Multipliers; but let us ſuppoſe in the moſt common Caſks, the Bung-Diameter to be 32 Inches, and Head 26, then for the Fruſtum of a

*Spheroid*
*Parab. Spindle*
*Parab. Con.*
*Cone*

$$m' = \left\{ \begin{array}{l} \frac{2}{3} + \frac{1}{3} \times \frac{6}{190} = ,6888, \&c. \\ \frac{2}{3} + \frac{4}{15} \times \frac{6}{186} = ,6753, \&c. \\ \frac{1}{2} + \frac{1}{4} \times \frac{6}{58} = ,5259, \&c. \\ \frac{1}{2} + \frac{1}{12} \times \frac{6}{58} = ,5087, \&c. \end{array} \right\} \textit{nearly.}$$

Which will ſerve indifferently, when the Bung-Diameter is betwixt *four* and *five* times the Difference betwixt it and the Head.

But thoſe (,7 ; ,68 ; ,54 ; ,52) given by Mr. *Bamford*, can never anſwer, unleſs where the Bung-Diameter is betwixt *three* and *four* times the Difference betwixt that and the Head.

From what has been ſaid, the *Limits* of all the Values of *m*, are eaſily deduced, by putting for *b*, its *extreme Values* in the ſecond Series of Equations ; now they are *b* and o, for 'tis evident *b* muſt

muſt ever be greater than the greateſt, and o leſs than the leaſt, both which being ſubſtituted for $b$, we have

$$\frac{o}{b} = \left\{ \begin{array}{c} \frac{6m-2}{3m^2-1} \\ \frac{30m-10}{15m^2-3} \\ \frac{4m-3}{2m^2-1} \\ \frac{6m-3}{3m^2-2} \end{array} \right\} \text{ and } \left\{ \begin{array}{c} \frac{6m-2}{3m^2-1} \\ \frac{30m-10}{15m^2-3} \\ \frac{4m-3}{2m^2-1} \\ \frac{6m-3}{3m^2-2} \end{array} \right\} = 1$$

And finding $m$ in theſe Equations, 'tis manifeſt all Values thereof for the reſpective Varieties *Pag.* 164, muſt be

Betwixt $\left\{ \begin{array}{l} \frac{1}{3} = ,3333, \&c. \\ \frac{1}{2} = ,5 \end{array} \right.$ and $\left\{ \begin{array}{l} \frac{1}{3} = ,3333, \&c. \\ \frac{1}{2} = ,5 \end{array} \right.$

and $\left\{ \begin{array}{l} 1 - \sqrt{\frac{2}{3}} = ,1835 \\ 1 - \sqrt{\frac{8}{15}} = ,2697 \\ 1 - \sqrt{\frac{1}{2}} = ,2927 \\ 1 - \sqrt{\frac{1}{3}} = ,5774 \end{array} \right.$ &c.

And

And theſe Limits taken from *Unity*, leave thoſe of $m'$, which therefore are

Bewixt $\left\{\begin{array}{l}\frac{2}{3}=,6666, \&c.\\ \frac{2}{3}=,6666, \&c.\\ \frac{1}{2}=,5\\ \frac{1}{2}=,5\end{array}\right\}$ and $\left\{\begin{array}{l}\sqrt{\frac{2}{3}}=,8165\\ \sqrt{\frac{8}{15}}=,7303\\ \sqrt{\frac{1}{2}}=,7071\\ \sqrt{\frac{1}{3}}=,5774\end{array}\right\}$ *&c.*

As they were firſt of all determin'd by Mr. *Simpſon*, in his Treatiſe above mention'd.

From the foregoing Theorems the Reader will be able, with very little trouble, to find a *true* Multiplier for reducing any of the Forms above into a Cylinder of equal Length; but as I have ſhewn how they may be meaſur'd by the Lines C, D, at *one ſetting of the Sliding-Rule*, there can never be occaſion for ſuch Reductions: nay, I can always gauge any of thoſe Caſks ſooner by thoſe Rules, than by firſt reducing them to Cylinders. So that theſe Rules are really of no Service in gauging a *full Cask*, but will be of particular Uſe when we come to *Ullaging*, which is conſider'd in the next Part.

CHAP.

# CHAP. V.

## *Of meaſuring the* Hoofs *of Pyramidic Cones.*

### DEFINITION I.

IF a Point V (*Fig.* 51.) be ſituate in a fixt Poſition above the Plane ACEDB (of any Figure whatever) and from V as a Pole an Indefinite right Line be carried round the ſame, it will deſcribe the *Likeneſs* of a Solid, which we call a *Pyramidic Cone*; of which ACEDB is the *Baſe*, and V its *Vertex*.

*Defin.* II. If a pyramidic Cone be cut by a Plane parallel to its Baſe, the Part contain'd betwixt the ſaid Plane and Baſe, is call'd its *Fruſtum*.

*Defin.* III. If a Plane be drawn thro' the Vertex of the Cone perpendicular to its Baſe, its Section with the Fruſtum is call'd the *vertical Plane* thereof; and then its Sections with the *End* and *Baſe*, their *Diameters* reſpectively.

*Defin.* IV. If a Plane be drawn through either Extremity of the End's Diameter, at right Angles to the vertical Plane of the Fruſtum, and ſo as to cut the Cone's Baſe, the Part of the Fruſtum contain'd betwixt the cutting Plane and Part of the Baſe, is call'd a *Hoof* thereof; and the remaining Part of the Fruſtum, the *complemental Hoof*; alſo that Extremity of the End's Diameter through which the Cutting-Plane paſſeth, may be call'd the *Vertex of the Hoof*.

*Defin.* V. The *Containing* Part of the Cone's Baſe, is call'd the *Baſe of the Hoof*; the Section of the Cutting-Plane and Fruſtum, its *Side*; and the Section of the ſaid Side and vertical Plane, the *Axis* of the Side.

LEMMA

## LEMMA I.

*If a Pyramidic Cone be cut by a Plane parallel to its Baſe, the Area of the Section will be to that of the Cone's Baſe, as the Square of the Perpendicular from the Vertex to the Plane of the Section, is to the Square of the Perpendicular's Height of the Cone.*

For let VEQ be the vertical Plane of the Cone, which cuts the Baſe in EQ, and the Section *aeq* in *eq*, then EQ and *eq* are the Diameters of the Baſe and Section *aeq*, (*Definit.* 3.) on which from V let fall the Perpendicular VP, meeting the firſt in P, and the other in *p*, then V*p* is the vertical Diſtance of the Section. Alſo in EQ aſſume any Point F, and ſuppoſe the Diſtance EF divided into an infinite Number of equal Parts in the Points G, H, K, *&c.* thro' every one of which and the Vertex V, let Planes be ſuppoſed drawn perpendicular to VEQ, cutting the Baſe in the Lines CD, MM, NN, OO, *&c.* and the Section *aeq* in the ſame manner in *cd*, *mm*, *nn*, *oo*, *&c.* thereby diſtinguiſhing each into an infinite Number of Elementa; let thoſe of the Baſe from F towards E be denoted by E′, E″, E‴, *&c.* and the correſponding ones of the Section *aeq* *e′*, *e″*, *e‴*, *&c.* then (*per Fig.*) E′=CD×FG and *e′*=*cd*×*fg*, but FG : *fg* :: FE : *fe* :: FP : *fp*, and FP : *fp* :: VP : V*p*; hence E′ : *e′* :: CD× VP : *cd*× V*p*. Again, (*from the Fig.*) CD : *cd* :: VP : V*p*, therefore E′ : *e′* :: $\overline{VP}^2$ : $\overline{Vp}^2$; and for the ſame Reaſon E″ : *e″* :: $\overline{VP}^2$ : $\overline{Vp}^2$; alſo E‴ : *e‴* :: $\overline{VP}^2$ : $\overline{Vp}^2$; whence (by the 12 *Eucl.* 5.) E′+E″+E‴, *&c.* *e′*+*e″*+E‴, *&c.* :: $\overline{VP}^2$ : $\overline{Vp}^2$ : But E′+E″+E‴, *&c.* =CED, and *e′*+ *e″*+*e‴*, *&c.* =*ced*, therefore CED : *ced* :: $\overline{VP}^2$ : $\overline{VP}^2$. *Q. E. D.*

*Cor.*

*Cor.* Whence it is evident all ſimilar Planes are to another, as the Squares of their homologous Lines.

## LEMMA II.

*The Meaſure of every Pyramidic Cone is equal to the Product of the Meaſure of its Baſe, by one third Part of its perpendicular Height.*

For let the Meaſure of the Cone's Baſe be $b$, its perpendicular Height VP$=h$, and $x$ the Meaſure of V$p$ (ſee *Figure* 51.) then by the preceding Lemma $h^2 : b :: x^2 : \frac{bx^2}{h^2}$, the Meaſure of the Section $acedb$. Further, let us imagine V$p$ divided into an infinite Number of equal Parts, every one of which is meaſured by $q$, and through each of them let Planes be ſuppoſed drawn parallel to the Baſe of the Cone, thereby dividing the Part V$aeb$ into an infinite Number of cylindrical Elementa, and let their ſucceſſive Diſtances from V be denoted by $x$, $x'$, $x''$, $x'''$, &c. *viz.* $x-q=x'$, $x'-q=x''$, $x''-q=x'''$, &c. then from what was ſaid above, the ſucceſſive Meaſures of the Baſes of theſe elementary Cylinders will be $\frac{bx^2}{h^2}$, $\frac{bx'^2}{h^2}$, $\frac{bx''^2}{h^2}$, $\frac{bx'''^2}{h^2}$, &c. and therefore the Meaſures of the Cylinders themſelves are $\frac{bx^2}{h^2} \times q$, $\frac{bx'^2}{h^2} \times q$, $\frac{bx''^2}{h^2} \times q$, $\frac{bx'''^2}{h^2} \times q$, &c. whence the Meaſure of the Part V$abc$, is equal to the Sum of an infinite Series of Squares, whoſe Roots, $x$, $x'$, $x''$, $x'''$, &c. are in Arithmetic Progreſſion from $o$ to $x$ incluſive, multiplied by $\frac{b \times q}{h^2}$, but the Number of them is $\frac{x}{q}$, whence (by the Lemma

*pag.* 85.) V*abc* is equal to $\frac{b \times q}{h^2} \times x^2 \times \frac{x}{q} \times \frac{1}{3} = \frac{bx3}{3h^2}$, which when $x$ becomes $h$, gives VAEB$=b \times \frac{h}{3}$. *Q. E. D.*

## Lemma III.

*Let* GHI (Fig. 52.) *be any Right-lined Triangle, and in* HG *a Point* E *taken, and the Line* IE *drawn; alſo through* E *let* EK *be drawn parallel to* IG, *and made equal to* EI, *and through the Point* K *a Line drawn parallel to* HI *meeting* GH *produc'd in* V : *then a Perpendicular let fall thence on* IE (*continued if needful*) *will be equal to a Perpendicular drawn from* H *on* IG, *that is* VF=HP.

For draw IQ perpendicular to HG, then from the ſimilar Triangles HGP, IGQ, we have GH : GI : : HP : IQ, but from the ſimilar Triangles HIG, VKE, GH : GI : : EV : EK = EI (*per Conſtr.*) hence HP : QI : : EV : EI. Again, from the Triangles EIQ, EVF, we have IQ : IE : : VF : VE ; whence, and the laſt Proportion, VF : VE : : HP : VE, therefore VF=HP. *Q. E. D.*

## Prop. I.

*Let* KEBA (Fig. 53.) *be the Fruſtum of a Pyramidic Cone, whoſe Pole is* V, *Baſe the Plane* AMBN, *End the Plane* KE, *Vertical Plane* VAB, *and* MNEB *a Hoof thereof : then* AB, KE *are the Diameters of the Baſe and End reſpectively, and* EL *the Axis of the Side ; on which make* EI *equal to* EK, *and through* I *draw* IG, IH *parallel to* AB, AV *reſpectively ; and from the Points* V, H, *let fall* VC, HP *Perpendiculars on* AB, IG ; *then* (*calling the Baſe of the Hoof* MNB, S, *and its Side* EMN, S′) *the Meaſure of the Hoof* MNEB *is expreſſed by this*

Theorem,

THEOREM, *viz.*

$$MNEB = \overline{BC \times S - GP \times S'} \times \frac{ER}{3BR}.$$

For through V and the Line MN, imagine a Plane drawn, whoſe Section with the Cone is the Triangle VMN, and from V let fall a Perpendicular on the Axis of the Side (produc'd if neceſſary) meeting the ſame in F.

Now 'tis manifeſt the Hoof MNEB is equal to the pyramidic Cone VMNB, made leſs by VMNE, but (by *Lemma* 2.) $VMNB = S \times \frac{VC}{3}$, and $VMNE = S' \times \frac{VF}{3}$. But $VC = \frac{ER \times BC}{BR}$ (for BR : ER :: BC : VC) alſo VF=HP by *Lemma* 3, and $HP = \frac{ER \times GP}{BR}$ (for BR : ER :: GP : HP) therefore for VC and VF in the above Values of VMNB and VMNE put their equals, and we have

$$MNEB = \overline{BC \times S - GP \times S'} \times \frac{ER}{3BR}. \quad Q.\ E.\ D.$$

*Cor.* 1. But if the Cone be upright, or the Triangle VAB iſoſceles, then $BC = \frac{1}{2}AB$, and $GP = \frac{1}{2}IP$; whence if D be put to denote the Diameter of the Baſe, and *d* that of the End of the Fruſtum, *viz.* D=AB, *d*=KE, and ER=*h*, the Height of the Fruſtum, then $BC = \frac{D}{2}$, $GP = \frac{d \times BL}{2EL}$, and $BR = \frac{D-d}{2}$; whence, in this Caſe, the Meaſure of the Hoof MNEB is expreſs'd by this

THEOREM, *viz.* $$\overline{D \times S - \frac{d \times BL}{EL} \times S'} \times \frac{h}{3 \times D-d}.$$

*Cor.*

*Cor.* 2. Let M β β N *k* ε ε *k* (*Fig.* 53.) be the Fruſtum of an upright ſquare Pyramid, which is cut through the Sides ε ε, MN by a Plane ε ε MN; then if D be the Side of the Baſe, *d* that of the End, and *h* the perpendicular Height of the Fruſtum, the Meaſure of the

$$\left\{\begin{array}{ll} \textit{Hoof} & \varepsilon\varepsilon \mathrm{MN}\,\beta\beta = 2\overline{\mathrm{D}+d} \times \frac{h\,\mathrm{D}}{6} \\ \textit{Com. Hoof} & \varepsilon\varepsilon \mathrm{MN}\,kk = \overline{2\,d+\mathrm{D}} \times \frac{h\,d}{6} \end{array}\right\}$$

For ſince the (Pyramidic Cone, or) Pyramid is upright, (by *Coroll.* 1.) we have $\varepsilon\varepsilon \mathrm{MN}\,\beta\beta = \overline{\mathrm{D} \times \mathrm{S} - \frac{d \times \mathrm{BL}}{\mathrm{EL}} \times \mathrm{S}' \times \frac{h}{3 \cdot \mathrm{D}-d}}$, but $\mathrm{S} = \mathrm{D}^2$, and $\mathrm{S}' = \frac{\mathrm{D}+d}{2} \times \mathrm{EL}$, as is evident from the Figure, whence $\frac{d \times \mathrm{BL}}{\mathrm{EL}} \times \mathrm{S}' = \frac{d \times \mathrm{D}}{\mathrm{EL}} \times \frac{\mathrm{D}+d}{2} \times \mathrm{EL} = d\,\mathrm{D} \times \frac{\mathrm{D}+d}{2}$, therefore $\overline{\mathrm{D} \times \mathrm{S} - \frac{d \times \mathrm{BL}}{\mathrm{EL}} \times \mathrm{S}' \times \frac{h}{3.\mathrm{D}-d}}$ becomes $\overline{\mathrm{D}^3 - \frac{1}{2}\mathrm{D}d \times \mathrm{D}+d} \times \frac{h}{3 \times \mathrm{D}-d} = \frac{2\mathrm{D}^3 - \mathrm{D}^2 d - \mathrm{D}d^2}{\mathrm{D}-d} \times \frac{h}{6}$, but $\frac{2\mathrm{D}^3 - \mathrm{D}^2 d - \mathrm{D}d^2}{\mathrm{D}-d} = 2\mathrm{D}^2 + \mathrm{D}\,d$, hence $\varepsilon\varepsilon \mathrm{MN}\,\beta\beta = 2\overline{\mathrm{D}+d} \times \mathrm{D} \times \frac{h}{6}$: and this taken from $\overline{\mathrm{D}^2 + \mathrm{D}d + d^2} \times \frac{h}{3}$, the Area of the whole Fruſtum (as eaſily follows from *Lemma* 2.) leaves $\varepsilon\varepsilon \mathrm{MN}\,kk = \overline{2d+\mathrm{D}} \times d \times \frac{h}{6}$. Q. E. O.

*And from hence theſe Rules, to meaſure the Hoofs of the Fruſtum of a ſquare Pyramid.*

RULE

### RULE I.

To twice the Side of the Baſe add that of the End, multiply the Sum by the Side of the Baſe, and that by one ſixth of the Height of the Hoof, the Product will be its Content.

### RULE II.

To twice the Side of the End add that of the Baſe, the Sum multiplied by the Side of the End, and that Product by one ſixth of the Fruſtum's Height, gives the Content of the *Complemental Hoof* By the SLIDING-RULE.

### RULE III.

Set 3,464 on D to the Fruſtum's Height on C, and find the Side of the Baſe, that of the End, and their Difference on D, noting the three Numbers oppoſite them on C. Then to the ſecond add five times the firſt, and from that Sum take the third, the Remainder is the Meaſure of the *Hoof*.

But if to the firſt five times the ſecond be added, and the third taken from that Sum, the Remainder is the Meaſure of the *Complemental Hoof*.

*Example.* Let there be the Fruſtum of a ſquare Pyramid, the Side of whoſe Baſe is 25, that of its End 16, and the Height of the Fruſtum 12 Inches, which is cut by a Plane through oppoſite Sides of its End and Baſe, to find the Meaſures of both the Hoofs.

OPERATION.

OPERATION.

$$\left\{\begin{matrix} D=25 \\ d=16 \\ h=12 \end{matrix}\right\}$$

| | |
|---|---|
| $50=2D$ | $32=2d$ |
| $16=d$ | $25=D$ |
| $66=2D+d$ | $57=2d+D$ |
| $25$ | $16=D$ |
| $330$ | $342$ |
| $132$ | $57$ |
| $1650=\overline{2D+d}\times D$ | $912=\overline{2d+D}\times d$ |
| $2=\frac{1}{6}h$ | $2=\frac{1}{6}h$ |
| $3300=\overline{2D+d}\times D\times\frac{1}{6}h$ | $1824=\overline{2d+D}\times d\times\frac{1}{6}h$ |
| *The Meaſ. of the Hoof.* | *The Meaſ. of the com. Hoof.* |

*Or*, 3,464 on D ſet to 12 on C, then againſt 25, 16, and 9 on D is 625, 256 and 81 on C reſpectively; therefore by the Rule,

$$\left\{\begin{matrix} 3125\ (=625\times 5)+256-81=3300 \\ 1280\ (=256\times 5)+625-81=1824 \end{matrix}\right\}\ \textit{as before.}$$

*Cor.* 3. If ABKE (*Fig.* 54.) be the Fruſtum of an Upright Cone with a circular Baſe, whoſe vertical Plane is ABKE, which is cut by a Plane thro' the Points E, A, and D be the Diameter of the Baſe, $d$ that of the End, and $h$ its Height, then the Meaſure of

$$\text{The}\ \left\{\begin{matrix} \textit{Hoof}\ \mathbf{ABE} \\ \textit{Com. Hoof.}\ \mathbf{AEK} \end{matrix}\right\} = \left|\begin{matrix} \overline{D^2-d\sqrt{Dd}}\times\frac{D\times h\times a}{3\times D-d}, \\ \overline{D\sqrt{Da}-d^2}\times\frac{d\times h\times a}{3\times D-d}. \end{matrix}\right.$$

For

For here the Points M, N and L, are united in A, and the Space S′ (by *Defin.* 3. *Chap.* 2.) is an Ellipſe, whoſe Tranſverſe is EA or EL, and Conjugate a mean Proportional betwixt D, *d*; hence S′= $EL \times \sqrt{Dd} \times a$, (*Cor.* 3. *Prop.* 4. *Chap.* 3.) and S= $D^2 \times a$. Put theſe Values for S and S′ in the *Theorem Cor.* 1. then the Meaſure of the Hoof is $D \times D^2 \times a - \frac{d \times D \times EL \times \sqrt{Dd} \times a}{EL} \times \frac{h}{3 \times D - d} = \overline{D^2 - d\sqrt{Dd}} \times \frac{D \times h \times a}{3 \times D - d}$; which taken from $\overline{D^3 - d^3} \times \frac{ah}{3 \times D - d}$, the Meaſure of the whole Fruſtum, (ſee *Cor.* 3. *Prop.* 2. *Chap.* 3.) leaves $\overline{D\sqrt{Dd} - d^2} \times \frac{d \times h \times a}{3 \times D - d}$, for the Meaſure of the *Complemental Hoof*. *Q. E. O.*

And from hence we have theſe two *Rules for meaſuring the elliptic Hoofs of an upright Cone.*

## RULE I.

From the Square of the Diameter of the Baſe, ſubſtract the Product of the Diameter of the End by a mean Proportional betwixt the two Diameters; multiply the Remainder by the Diameter of the Baſe, and that Product by the Height of the Fruſtum, and this by ,261799388 (or in Practice by ,2618) then if the laſt Product be divided by the Difference of the Diameters, the Quotient will be the Meaſure of the *Hoof* propoſed.

## RULE II.

From the Product of the Diameter of the Baſe by a mean Proportional betwixt that and the Diameter of the End, ſubſtract the Square

of the Diameter of the End, multiply the Remainder by the Side of the ſaid Square, and that Product by the Height of the Fruſtum, and this by ,261799388 (or ,2618) then if the laſt Reſult is divided by the Difference of the two Diameters, the Quotient will be the Meaſure of the *Complemental Hoof.*

*Examp.* Let the Diameter of the Baſe be 60, that of the End 38,4, and the Height 36 Inches,

OPERATION *for* ABE (*Fig.* 55.)

$D = \ldots 60$
$d = \ldots 38{,}4$

$D \times d = 2304{,}0$ $(48 = \sqrt{Dd}) = D - d = 21{,}6$

16 — 38,4 = *d* — 216

8) 704 — 3072 — 1634 — 3794688 — 275680,00
704 — 1536 — 1512 — 4997162,0

0 — 1843,2 = $dv\overline{Dd}$ — 1226 — 3513600
3600 = $D^2$ — 1080 — 1054080
17568
12298
1581
158
7

1756,8 = $D^2 - dv\overline{Dd}$ — 1468 — 45992,92 =
60 = D — 1296 — *Hoof* ABE.

105408,0 = $D^2 - dv\overline{Dd} \times D$ — 1728
36 = *b* — 1728

632448 — 0000
316224

3794688 = $D^2 - dv\overline{Dd} \times D \times b$

OPE-

OPERATION *for* AKE (*Fig.* 55.)

38,4 = *d*
384

1536
3072
1152

1474,56 = 1474,56 = $d^2$

48 = $\sqrt{Dd}$
60 = D
2880 = $D\sqrt{Dd}$
1474,56 = $d^2$
1405,4 = $D\sqrt{Dd}-d^2$
38,4 = *d*

562176
1124352
421632

53968,896 = $d \times \overline{D\sqrt{Dd}-d^2}$
36 = *b*

323813376
161906688

D—*d*=21,6) 1942880,256 | 89948,16
4997162

2148
2048
1040
1762
345
1296
0000

17989632
5396890
89948
62964
8095
810
36

23548,375 = *Comp. Hoof* AKE

We might alſo ſhew how to meaſure theſe Hoofs by the *Sliding-Rule*. But the Operation would be too tedious for Practice, ſo I proceed to the *Parabolic Hoof*.

*Cor.* 4. The ſame being ſuppoſed as in the laſt Corollary; only here let EL (*Fig.* 56.) the Axis of the Hoof's Side, be parallel to KA; then MEN the Side it ſelf (by *Defin.* 6. *Chap.* 2.) is a Parabola, and therefore the Meaſure of the

$$\left.\begin{matrix}\textit{Com. Hoof}\\ \textit{Hoof EBL}\end{matrix}\right\} = \left\{\begin{matrix}\frac{D}{D-d}\times S-\frac{4}{3}d\sqrt{d\times D-d}\times\frac{b}{3}\\ \hline D^2+d\times D+d\times a-\frac{D\times S}{D-d}+\frac{4}{3}d\sqrt{d\times D-d}\times\frac{b}{3}\end{matrix}\right.$$

For

For S′ (by *Prop.* 4. *Chap.* 3.) $= \frac{2EL \times MN}{3} = \frac{4EL}{3}\sqrt{d \times D-d}$, therefore $\frac{d \times BL}{EL} \times S' = d \times BL \times \frac{4}{3}\sqrt{d \times D-d}$, or $\frac{4}{3} \times d \times D-d\sqrt{d \times D-d}$; let this be put in its Room in the Theorem *Cor.* 1. and it becomes $D \times S - \frac{4}{3} \times d \times D-d\sqrt{d \times D-d} \times \frac{h}{3 \times D-d} = \frac{D}{D-d} \times S - \frac{4d}{3}\sqrt{d \times D-d} \times \frac{h}{3}$, and this taken from $D^2+d \times D+d \times \frac{ah}{3}$ the whole Fruſtum, leaves the Meaſure of the *Complemental Hoof* as above. *Q. E. O.*

*Exampl.* Let the Dimenſions of a Conic Fruſtum be as above, to find the Meaſure of the *Parabolic Hoof.*

Firſt we muſt find S the Segment of a Circle, whoſe Diameter is D=60, and verſed Sine D—*d* =21,6: Whence by the Rule, *Page* 105, I annex two Cyphers to 21,6, that ſo it may conſiſt of three Decimals as the Rule requires, and it becomes 21,600, which divided by 60, gives ,360, againſt which in *the Table of Segments* is ,2545505; this multiplied by 3600, the Square of 60, gives 916,3819=S, alſo $\frac{D}{D-d} = \frac{100}{36}$, therefore $\frac{D}{D-d} \times S = \frac{100S}{36}$, and from hence the following

OPERATION *for* ELB, *Fig.* 56.

$d = 38.4$

$D-d = 21.6$

2304
384
768

$\sqrt{d \times D-d} = \sqrt{829,44} = 28,8$

4

4) 429
384

56) 4544
4544
0000

576
288
1440

$51,2 = \frac{4}{3}\frac{d}{}$

$1474,56 = \frac{4}{3}\sqrt{D \times D-}$

36) 91638,19
196
163
198
181
190

$2545,5053 = \frac{100S}{36}$

1474,56

1070,9453
12

12851,3436

$= \frac{b}{3}$

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Hence

Hence the Meaſure of the *Parabolic Hoof* is 12851,3436; and this taken from $\overline{D^2+d\times D+d} \times \frac{a\,b}{3} = 69541{,}2893$, the Meaſure of the whole Hoof, leaves 56689,9457 for the *Complemental Hoof* ELB.

*Cor.* 5. The ſame things being ſtill ſuppoſed, only here let MEN (*Fig.* 57.) be an Hyperbola, then the *Meaſure of the Hyperbolic Hoof* MNEB, *is expreſſed by this*

THEOREM, *viz.*

$$\text{MNEB} = D \times S - \frac{d \times LB}{LE} \times S' \times \frac{b}{3 \times \overline{D-d}}.$$

Where LE is the Diameter of the Side, and LB that of the Baſe, which muſt be given before theſe Hoofs can be meaſured: But if MEN, the Cutting Plane, ſhould be Perpendicular to the Baſe of the Cone, then the *Meaſure of the Hoof* MNEB *is expreſſed by this*

THEOREM, *viz.*

$$\text{MNEB} = \frac{D\,b}{3 \times \overline{D-d}} \times S - \frac{d}{6} \times S'.$$

And the tranſverſe Axis of the Hyperbola, $S'$ or MEN, is $\frac{2\,d\,b}{D-d}$, Conjugate $\frac{d}{2}$.

The firſt Theorem is evident from *Corol.* 1, and the ſecond follows from that; for when MNE is perpendicular to the Baſe of the Cone, then $LB = \frac{D-d}{2}$, and $LE = b$, which being put for LB and LE, will give the Theorem above.

But as $\frac{D-d}{2}$ is commonly ſmall in reſpect of D, we may for thoſe Caſes give an *Approximation* of

*Theorem* 2, which is very near the Truth; for D, *d*, and *h* ſtanding for the ſame Sines as before,

$$MNEB = \frac{D \times 47D + 23d}{8D + 6d} - \frac{d \times 47d + 23D}{8d + 6D} \times \frac{h}{45}\sqrt{D^2 - d^2} \text{ proximè.}$$

*For an Example* to theſe two Theorems, let $D = 32$, $d = 24$ and $h = 20$, therefore $BL = \frac{D-d}{2} = 4$.

In the firſt place we muſt find the Values of S and S′, that is the Area of the Segment of a Circle whoſe Diameter is 32, and verſed Sine 4; alſo the Area of an Hyperbola, whoſe tranſverſe Diameter is $\frac{2dh}{D-d} = 120$, Conjugate 12, and Abſciſſa 20; for the firſt I annex three Cyphers to 4, and devide it by 32, the Quotient 125 I look for in the Table of Segments under VS, againſt which under *Seg.* is ,0566640; this multiplied by $\overline{32}^2 = 1024$ gives 58.0239=S. And to find S′, (by *Prop.* 6. *Chap.* 3.) we muſt put for *x*, *y*, and V their Values, which here are 20, $\sqrt{4 \times 28}$, and 140; therefore calling $\frac{4yx}{3} = \frac{80}{3}\sqrt{112} = 282,21347 = A$; then $S' = A - \frac{A}{5 \cdot 7} - \frac{B}{7 \cdot 7} - \frac{3C}{9 \cdot 7} - \frac{5D}{11 \cdot 7}$, *&c.* as follows.

| | |
|---|---|
| A = | 282.21347 |
| B = | 8,06324 |
| C = | 16456 |
| D = | 784 |
| E = | 51 |
| F = | 5 |
| Sum — | 8,23620 |
| S′ = | 273,97727 |

There

Therefore for S and S′, put theſe Values in the ſecond Theorem, then $\frac{Dh}{3 \times D - d} \times S$, becomes $\frac{32 \times 20}{3 \times 8}$

$\times 58{,}0239 =$ . . . . 1547,3040

and $\frac{d}{6} \times S' = 4 \times 273{,}97727 =$ 1095,9091

Their Difference is the Meaſure of EMNB = . . . 451,3949

But if we uſe the Approximation, then we ſhall have

$$32 \times \frac{47 \times 32 + 23 \times 24}{8 \times 32 + 6 \times 24} - 24 \times \frac{47 \times 24 + 23 \times 32}{8 \times 24 + 6 \times 32} \times \frac{20}{45}\sqrt{448}$$

that is EMNB $= 164{,}48 - 116{,}5 \times \frac{4}{9}\sqrt{448} =$ 451,35, nearly the ſame as before.

So that from what has been ſaid in this Propoſition and its Corollaries, the Reader will be able to meaſure the Hoof of any Cone, let the Figures of the Baſe and Side be what they will; from whence is deduced the Method of Gauging an inclin'd Conical Tun. Alſo from the laſt Corollary any Caſk in the Shape of two Fruſtums of a Cone may be ullaged; but in what follows, I ſhall lay down a Propoſition for meaſuring Planes or Solids by Approximation, a thing of the greateſt Importance to this Part of Science, of any that was ever brought for that Purpoſe, ſince it may be ſaid to contain the *whole Art of Gauging*, and that of *Coppers*, *Stills*, *Tuns*, as well as all kinds of Caſks, whether full or part empty, either ſtanding or lying.

CHAP.

## CHAP. VI.

### *Of meaſuring Curve-Lin'd Planes, and Solids, by* Approximation.

PROP.

IF MQ=$y'$, NR=$y''$, PS=$y'''$ (Fig. 58.) *repreſent three equidiſtant perpendicular Ordinates to the Axis of a Curve* MNP, *whoſe Equation is* $y=a+bx+cx^2$, *where* x *ſtands for any Abſciſs* QT, *and* y *its Ordinate* TO; *then, calling* QS *the Diſtance of the extream Ordinates* $l$, *the Meaſure of the Space* QMPS *ſhall be expreſs'd by this*

THEOREM, *viz.* $QMPS=\overline{y'+y'''+4y''}\times\frac{l}{6}$.

For by the preceding Principles, the Quadrature of the Curve, whoſe Abſciſſa is $x$, and Ordinate $a+bx+cx^2$, will be found $\overline{a+\frac{bx}{2}+\frac{cx^2}{3}}\times x$; and this when $x$ becomes $l$, is $\overline{a+\frac{bl}{2}+\frac{cl^2}{3}}\times l = \overline{6a+3bl+2cl^2}\times\frac{l}{6}$, = QMPS.

But from the Equation of the Curve we have theſe three Equations,
$$\left\{\begin{array}{l} y'=a \\ y''=a+\frac{bl}{2}+\frac{cl^2}{4} \\ y'''=a+bl+cl^2 \end{array}\right.$$

And by taking the Difference of thoſe above, we have theſe,
$$\left\{\begin{array}{l} y''-a=\frac{bl}{2}+\frac{cl^2}{4} \\ y'''-y''=\frac{bl}{2}+\frac{3cl^2}{4} \end{array}\right.$$

And the Difference of the laſt gives
$$\left\{y'''-2y''+a=\frac{cl^2}{2}\right.$$

Whence

Whence $2cl^2 = 4 \times \overline{y''' - 2y'' + a}$; and by the 3d Equation above, $bl = y''' - a - cl^2$, therefore $3bl = 3y''' - 3a - 6 \times \overline{y''' - 2y'' + a} = -3y''' + 12y'' - 9a$; put theſe for $2cl^2$ and $3bl$, in the above Expreſſion of the Area, and then we have

$$QMPS = \overline{6a + 3bl + 2cl^2} \times \frac{l}{6} = \overline{y' + 4y'' + y'''} \times \frac{l}{6}.$$

*Q. E. D.*

*Cor.* 1. The ſame Method of Demonſtration extends to any Number of equidiſtant Ordinates; ſo if A denotes the Sum of the extream Ordinates, B the Sum of thoſe next to them, C the Sum of the two next following the laſt, and ſo on; then we ſhall have the following *Tables of Areas*, for the ſeveral Numbers of Ordinates prefixt againſt them, *viz.* for

| | |
|---|---|
| 2 | $A \times \frac{l}{2}$ |
| 3 | $\overline{A + 4B} \times \frac{l}{6}$ |
| 4 | $\overline{A + 3B} \times \frac{l}{8}$ |
| 5 | $\overline{7A + 32B + 12C} \times \frac{l}{90}$ |
| 6 | $\overline{19A + 75B + 50C} \times \frac{l}{288}$ |
| 7 | $\overline{41A + 216B + 27C + 272D} \times \frac{l}{840}$ |
| 8 | $\overline{751A + 3577B + 1323C + 2989D} \times \frac{l}{17280}$ |
| &c. | &c. |

This Method was invented by Sir *Iſaac Newton*, and publiſhed by Mr. *Jones* in 1711; and ſince proſecuted by Mr. *Coats*, Mr. *De Moivre*, and by Mr. *Stirling*, in a whole Treatiſe entirely built thereon,

thereon, where ſuch as deſire a further Inſight into this matter, may find it ſufficiently explain'd, and applied to ſome of the moſt intricate Parts of Mathematicks.

But as I am going from thence to offer a new Method of Gauging, by which all *Gueſſing* will be laid aſide, I judg'd it not amiſs to ſhew its Uſe in ſome Inſtances of a different Nature.

And *Firſt*, 'Tis known the *Hyperbolic Logarithm of any Number denoted by* $1+x$, *is as the Area of a Curve whoſe Abſciſſa is* $x$ *and Ordinate* $\frac{1}{1+x}$; therefore to find this Area, or the Log. of $1+x$, let $x$ be ſuppoſed equal to 1, and divided into ſix equal Parts, then we ſhall have the ſeven following equidiſtant Ordinates $\frac{1}{1+0}$, $\frac{1}{1+\frac{1}{6}}$, $\frac{1}{1+\frac{2}{6}}$, $\frac{1}{1+\frac{3}{6}}$, $\frac{1}{1+\frac{4}{6}}$, $\frac{1}{1+\frac{5}{6}}$, $\frac{1}{1+\frac{6}{6}}$; that is, $\frac{6}{7}$, $\frac{6}{8}$, $\frac{6}{9}$, $\frac{6}{10}$, $\frac{6}{11}$, $\frac{6}{12}$, and therefore according to the Rule, againſt the Ordinates in the Table of Areas, we have this Proceſs:

A=

$A = 1 + \frac{6}{12} = \frac{18}{12} = 1,5$ -

$B = \frac{6}{7} + \frac{6}{11} = \frac{108}{77} = 1,4025974$

$C = \frac{6}{8} + \frac{6}{10} = \frac{27}{20} = 1,35$

$D = \frac{6}{9} + 0 = \frac{2}{3}$, 6666666

and

$41A = 61,5$

$216B = 302,9610384$

$27C = 36,45$

$272D = 181,3333333$

*Sum* $582,2443717$

And the Sum 582,2443717, divided by 840, equal to $\frac{l}{840}$ (for $l = x = 1$) gives ,6931479, which is the *Hyperbolic Log.* of $2 = 1 + x$, and would have required at leaſt 1000000 Terms of the common Series $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, *&c.*

For a *Second Example*; Let it be required to ſquare the Curve, whoſe Equation is $\frac{1}{1+t^2}$, this (by what was ſhewn *Prop.* 2. *Chap.* 1.) is equivalent to the circular Arch, whoſe Radius is 1, and Tangent $t$; let the Arch be 45 Degr. then $t = 1$,

$t=1$, which ſuppoſe divided into ſix equal Parts, and we have theſe ſeven equidiſtant Ordinates,

$$\frac{1}{1+0},\ \frac{1}{1+\frac{1}{36}},\ \frac{1}{1+\frac{4}{36}},\ \frac{1}{1+\frac{9}{36}},\ \frac{1}{1+\frac{16}{36}},\ \frac{1}{1+\frac{25}{36}},\ \frac{1}{1+\frac{36}{36}}$$

*Or*, $1.\frac{36}{37},\ \frac{36}{40},\ \frac{36}{45},\ \frac{36}{52},\ \frac{36}{61},\ \frac{36}{72}.$

And hence by the Rule againſt the Ordinates in the Table, this Proceſs,

$$\left.\begin{array}{l} A=1+\frac{36}{72}=1,5 \\ B=\frac{36}{37}+\frac{36}{61}=1,563137 \\ C=\frac{36}{40}+\frac{36}{52}=1,192308 \\ D=0+\frac{36}{45}=0,8 \end{array}\right\} \text{and} \left\{\begin{array}{l} 41A=61,5 \\ 216B=337,637592 \\ 27C=42,992316 \\ 272D=217,6 \end{array}\right.$$

*Sum*, 659,729908

And this Sum divided by 840, gives ,785393 for the Length of the circular Arch of 45 Deg. erring but 5 in the laſt place.

Theſe two Examples will be ſufficient to give the Reader a Proof of the great Uſe and Accuracy of this Method in Quadratures; I ſhall now proceed

ceed to ſhew its Application in meaſuring of Solids.

*Cor.* 2. For if $y'$, $y''$, $y'''$ in the Propoſition, inſtead of the Ordinates MQ, NR, PS, ſhould denote the Circles generated by them, when the whole Curve MNP turns on QS as an Axis, then the Expreſſion itſelf would be the Meaſure of the Solid, generated from that Motion, which therefore is expreſs'd by $\overline{y' + 4y'' + y'''} \times \frac{l}{6}$, or putting $d'$, $d''$, $d'''$ for the Diameters of thoſe Circles, the Meaſure of the Solid will be $\overline{d'^2 + 4d''^2 + d'''^2} \times \frac{al}{6}$.

And therefore when this Method is applied to meaſuring Solids, the Quantities A, B, C, D, *&c.* *Page* 187. denote Spaces, but Lines when uſed in meaſuring Spaces.

*Cor.* 3. So if $S'$ denote one End of a Solid, $S'''$ the other End, and $S''$ a Section thereof by a Plane in the Middle betwixt the two Ends, then the Meaſure of that Solid is nearly expreſs'd by $\overline{S' + 4S'' + S'''} \times \frac{l}{6}$, where $l$ denotes the perpendicular Diſtance betwixt the Planes $S'$, $S'''$.

The ſecond *Corol.* is ſufficient for Meaſuring or Gauging all Solids, generated by a Curve revolving on ſome Line as an Axis; and the third for all others, or Ullaging a lying Caſk.

*Cor.* 4. For let the Bung and Head-Diameters of any Caſk be denoted by $b$, $h$, and the Length $l$, as before; alſo put $m$ to denote a Diameter taken in the Middle betwixt the Bung and Head, then for meaſuring any Caſk whatever in cubic Inches, this is the

THEOREM, $\overline{b^2 + h^2 + 4m^2} \times \frac{al}{6}$, *nearly.*

Or,

Or *by the Sliding-Rule*, ſet 2,764 on D to the Length on C, and find the Bung, Head, and Middle Diameters on D, noting the three Numbers oppoſite them on C; then to the Sum of the firſt and ſecond add four times the third, and the Sum is the Meaſure ſought. How to obtain the ſame in Gallons will be ſhewn preſently.

To engage the Reader's Attention ſtill more in favour of this Method, I ſhall ſhew what Theorems are thence deducible, for meaſuring the ſeveral Varieties of Caſks, mention'd in the firſt *Scholium* of the laſt Chapter. For as the Forms of thoſe Solids are given, we can by help of the Equation, defining the Curve by whoſe Revolution they may be ſuppoſed generated, find *m* the *Middle Diameter*, which in Practice, when we do not know the Form, muſt be found by actual Menſuration.

*Cor.* 5. For if the Sides of a Solid were ſtreight, and the two Ends two Ellipſes, parallel and ſimilarly poſited, or ſo that the longeſt Axis of the one was parallel to the ſhorteſt of the other, and B, W denoted them of the greateſt End, alſo *b*, *w* thoſe of the leaſt reſpectively; then, if this Solid was cut through its Middle by a Plane, parallel to its Ends, the Section which is parallel to B, *b*, will be denoted by $\frac{B+b}{2}$, and that parallel to W, *w*, by $\frac{W+w}{2}$; whence the Area of the greater End is $B \times W \times a$, the leſſer $b \times w \times a$, and of the Section in the Middle $\frac{B+b}{2} \times \frac{W+w}{2} \times a$, (as is evident from *Cor.* 1. *Prop.* 5. *Chap.* 3.) If theſe are put for $S'$, $S'''$, and $S''$, in *Cor.* 3. foregoing, we have $\overline{B \times W + b \times w + \overline{B+b} \times \overline{W+w}} \times \frac{al}{6}$ for

for the Meaſure of this Solid, or $\overline{W \times 2B+b} + \overline{w \times 2b+B} \times \frac{al}{6}$, *which is a general Theorem for meaſuring all ſtreight-ſided Tuns, whoſe Ends are circular, elliptical or rectangular, and ſituate as above.* For let the two Ends be Circles, then WB and $w=b$; whence for meaſuring the Fruſtum of a right Line we have $\overline{2B^2+2b^2+2Bb} \times \frac{al}{6}$, or $\overline{B^2+b^2+Bb} \times \frac{al}{3}$, as was ſhewn *Cor.* 3. *Prop.* 2. *Chap.* 4.

If the two Ends are Squares, then $a$ in the laſt Theorem muſt be made equal to Unity, whence $\overline{B^2+b^2+Bb} \times \frac{l}{b}$, is the Meaſure of a Tun in the Shape of the Fruſtum of a ſquare Pyramid.

But if the Ends are Rectangles, then $\overline{W \times 2B+b} + \overline{w \times 2b+B} \times \frac{l}{6}$ is the Meaſure of all ſtreight-lined Tuns, whoſe Breadths at Top and Bottom are $b$, B, or B, $b$, and Widths correſponding $w$, W, or W, $w$.

We might from the ſame Principle deduce the Meaſures of Tuns, having their Ends of any other Figures, but they never occurring in Practice, I decline ſuch uſeleſs Purſuits, and proceed to

*Cor.* 6. If MMPP be *the Fruſtum of a parabolic Conoid*, the Diameter of whoſe greater End is $b$, leſſer $h$, and Length $l$, as before; alſo put $p$ the Parameter of the generating Parabola; then (by *Cor.* 3. *Prop.* 2. *Chap.* 2.) $p \times l = \frac{b+h}{2} \times \frac{b-h}{2}$, and $p \times \frac{l}{2} = \frac{b+m}{2} \times \frac{b-m}{2}$, whence $m^2 = \frac{b^2+h^2}{2}$,

*the Square of the middle Diameter*; therefore, by *Cor.* 4. foregoing, $\overline{b^2+h^2+4m^2} \times \frac{al}{6} = \overline{b^2+h^2} \times \frac{al}{2}$, as found before, *Page* 123.

*Cor.* 7. If MMPP repreſent the *Fruſtum of a Spheroid*, whoſe tranſverſe Axis is $t$, the reſt as above, then (from *Cor.* 2. *Prop.* 1. *Chap.* 2.) $t^2 : b^2 :: t^2-4l^2 : h^2$; and $t^2 : b^2 :: t^2-l^2 : m^2$, whence $m^2 = \frac{3b^2+h^2}{4}$, the Square of the Middle-Diameter, therefore now $\overline{b^2+h^2+4m^2} \times \frac{al}{6} = \overline{2b^2+h^2} \times \frac{al}{3}$, as was ſhewn *Page* 126.

Theſe three Caſes agree exactly with the Solutions given from other Principles, but in thoſe that follow, eſpecially the two laſt, which do not admit of a perfect Cubation; the Solutions found from this Method, will deviate a ſmall matter from the Truth, but never ſo much as to be of the leaſt Conſequence in Gauging.

*Cor.* 8. Let MMPP be the *Fruſtum of a parabolic Spindle*, the Letters $b$, $h$, $l$, $m$, denoting the ſame Lines as before; then from what was ſaid *Prop.* 5. *Chap.* 4. $h = b - \frac{2l^2}{p}$, and therefore $m = b - \frac{l^2}{2p} = \frac{3b+h}{4}$, *the Middle Diameter.* Put this for $m$ (*Cor.* 4.) and we have $\overline{2b^2+h^2-\frac{3}{8} \times \overline{b-h}^2} \times \frac{al}{3}$, too much by $\overline{b-h}^2 \times \frac{al}{120}$, which can never amount to $\frac{1}{100}$th of a Gallon in Gauging any Caſk to be met with in Practice. But if we ſhould ſuppoſe another Fruſtum exactly equal to this in every reſpect join'd to it; then we have the Value of five Equidiſtant Diameters of this Solid, the

the Extreme ones being $h$, $h$, the next to theſe $m$, $m$; and the middle one $b$; whence by the Rule againſt five Ordinates in the Table, we have $\sqrt{7A+32B+12C} \times \frac{al}{6}$, which is equivalent to $\sqrt{14b^2+\frac{32}{8} \times \overline{3b+h}^2+3b^2} \times \frac{al}{90}$; for $A=ab^2+ab^2=2ab^2$, $B=am^2+am^2=a \times \frac{\overline{3b+h}^2}{8}$, and $C=ab^2$: which put for A, B, C, gives the Expreſſion above; and that contracted, is equal to $\sqrt{2b^2+h^2-\frac{2}{5} \times \overline{b-h}^2} \times \frac{al}{3}$, the Meaſure of a parabolic Spindle, the Diameter of whoſe Ends are $h$, that in the Middle betwixt them $b$, and Length $l$; exactly the ſame as was ſhewn *Prop.* 5. *Chap.* 4.

*Cor.* 9. Laſtly, let MMPP repreſent the *Fruſtum of a Circular*, or *Equilateral Hyperbolic Spindle*, and $t$ the Diameter of the generating Curve, then from what was ſaid *Prop.* 6. *Chap.* 4. we have for the $\left\{\begin{matrix}\textit{Firſt}\\ \textit{Second}\end{matrix}\right\}$

$$t=\left\{\begin{matrix}\frac{1}{2} \times \frac{l^2}{b-h}-\frac{1}{2} \times \overline{b+h}+b\\ \frac{1}{2} \times \frac{l^2}{b-h}+\frac{1}{2} \times \overline{b+h}-b\end{matrix}\right\} \overset{\text{and}}{=} \left\{\begin{matrix}b-t+\sqrt{t^2-\frac{l^2}{4}}\\ b+t-\sqrt{t^2+\frac{l^2}{4}}\end{matrix}\right\}$$

But ſince in the Theorems (*Scholium* 2. *Ch.* 4.) for meaſuring theſe Solids, there is S a Segment of the generating Curve, therefore we can no otherwiſe ſhew the Coincidence of the Meaſure deduced from this Method, with the true Solution there given, but by a particular Example; for which purpoſe let $b=32$, $h=24$, and $l=40$, the ſame as in the above Scholium, then for the

$$\left.\begin{matrix}\textit{Circle}\\ \textit{Eq. Hyp.}\end{matrix}\right\} t = \left\{\begin{matrix}104\\ 96\end{matrix}\right\} m = \left\{\begin{matrix}30{,}059\\ 29{,}939\end{matrix}\right.$$

And therefore (by the Theorem *Corollary* 4.) $\overline{b^2 + h^2 + 4m^2} \times \frac{al}{6}$ gives for the

$$\left.\begin{matrix}\textit{Circle}\\ \textit{Eq. Hyp.}\end{matrix}\right\} = \left\{\begin{matrix}\textit{C. Inches.}\\ 27301{,}12\\ 27150{,}34\end{matrix}\right\} \text{or} \left\{\begin{matrix}\textit{A. Gall.}\\ 96{,}81\\ 96{,}28\end{matrix}\right.$$

The firſt differs but $\frac{5}{100} = \frac{1}{20}$th Parts of a Gallon from the Truth; and the Second but $\frac{6}{100} = \frac{1}{16\frac{2}{3}}$ds of a Gallon.

From what has been ſaid, it plainly appears by this Method, you are certain either to have the true Content, or within the 20th Part of a Gallon; whereas I may venture to affirm, by the common way of gueſſing at the Form of a Caſk, and thence finding its Content (by reducing it to a Cylinder from the ordinary Multipliers) you can never be ſure within two or three Gallons, or perhaps more; and this Method being alſo very ready in Practice, I would fain hope ſome time or other to ſee it generally follow'd by the Officers of Exciſe.

Having now laid down all the neceſſary Theorems for Gauging, I ſhall proceed to ſhew their Application to Practice.

PART

# PART III.

## *The Practice of Gauging.*

IF what has been ſaid in the preceding Pages was thoroughly conſidered, there would be no neceſſity of adding any thing further; for ſince

A { *Gallon of Ale* / *Gallon of Wine* / *Buſhel of Malt* } contains { 282 / 231 / 2150,42 } *Cub. Inch.*

'Tis manifeſt if the Meaſure of any Solid in Cubic Inches, found by the foregoing Rules, be divided by any of the Numbers above, the Quotient will reſpectively ſhew, how many Gallons of Ale, Wine, or Buſhels of Malt, a Veſſel in that Form will contain.

But as this Tract may fall into the Hands of ſome, whoſe Employment requires they ſhould be acquainted with the Practice of Gauging, and yet have not Time, or perhaps Patience to inform themſelves with the Theory, and ſo might have ſome Difficulty to fix on the proper Rules or Theorems for their purpoſe, as they are intermix'd with the inventive Part, and alſo the Theorems themſelves, require ſome farther elucidation to render them more eaſy to Learners; I have for their ſakes added this Third Part, which contains the Practice, or all the Rules for Gauging, neceſſary to be known by every Officer in the Exciſe.

## CHAP. I.

### *Of Cask-Gauging.*

THIS Chapter is divided into three Sections, containing three different Methods for Gauging a full Caſk. The *firſt* and *ſecond* are on Suppoſition that the Form of the Caſk is known, and alſo that it is generated by a Circle, or one of the Conic Sections turning round its Axis, or an Ordinate thereof. But the *third Method* is by taking a *fourth Dimenſion* without any regard to the Form, which is required in the other two.

The Number of Forms into which Caſks are commonly diſtinguiſh'd in Practice, are only three, *viz.* two Fruſtums of a Spheroid, two Fruſtums of a Parabolic Spindle, and two Fruſtums of a Cone; but as there undoubtedly may be others betwixt theſe, and alſo becauſe we have given Rules for Meaſuring ſome of the intermediate ones, we diſtinguiſh them in ſix Forms; the Order and Names of which are as follows:

| | | | |
|---|---|---|---|
| *Variety the* | 1 | *Two equal Fruſtums of a* | *Spheroid.* |
| | 2 | | *Circular Spindle.* |
| | 3 | | *Parabolic Spindle.* |
| | 4 | | *Eqilat. Hyp. Spin.* |
| | 5 | | *Parabolic Conoid.* |
| | 6 | | *Cone.* |

Some have laid down Rules to diſcover theſe Forms one from another by Inſpection, but as I conceive that impoſſible, inſtead thereof we refer the Reader to the Figures 58, 59, 60, 61, 62, 63, which are exactly drawn for the ſeveral Varieties above, when the Bung-Diameter is 32,

Head

Head 26, and Length 40 Inches; by which he will acquire a better Idea of the ſeveral Varieties above, than can poſſibly be convey'd by any verbal Deſcription whatever.

## SECTION I.

### PROP. I.

*The Length, Head, and Bung-Diameters of a Cask in the Shape of two equal Fruſtums of a* Spheroid *being given, to find its Meaſure in Ale or Wine-Gallons.*

*Solution.* To twice the Square of the Bung-Diameter, add the Square of that of the Head, and multiply the Sum by the Length of the Caſk; then if that Product is multiplied

by $\left\{\begin{matrix},0009284\\,0011333\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1077,15\\882,35\end{matrix}\right\}$ for $\left\{\begin{matrix}\textit{Ale},\\\textit{Wine},\end{matrix}\right.$

the Product, or Quotient, will be the Meaſure requir'd.

*Or by the Sliding-Rule.*

Set $\left\{\begin{matrix}32,82\\29,7\end{matrix}\right\}$ for $\left\{\begin{matrix}\textit{Ale}\\\textit{Wine}\end{matrix}\right\}$ on D, to the Caſk's Length on C, and on D find both the Bung and Head-Diameters, noting the two Numbers oppoſite them on C; then if the ſecond is added to twice the firſt, the Sum will be the Meaſure required.

*Example.* Let ABCD (*Fig.* 58.) repreſent a Caſk in the Shape of two equal Fruſtums of a Spheroid, whoſe Bung-Diameter EF is 32 Inches, Head BD=AC 26 Inches, Length GH 40 Inches, and its Meaſure requir'd in Ale and Wine Gallons.

| | | |
|---|---|---|
| Twice the Square of (32) the Bung- | Diam.s | 2048 |
| The Square of (26) the Head- | Diam.s | 676 |
| Their Sum | | 2724 |
| Multiplied by (40) the Length | | 40 |
| Gives | | 108960 |
| This multiplied by ,0009284 *inverted* | | 4829000,0 |
| | | 9806 |
| | | 218 |
| | | 87 |
| | | 4 |
| Gives the Content in *Ale Gallons* | | 101,15 |
| *Or*, | | 108960 |
| Multiplied by | | 3331100,0 |
| | | 10896 |
| | | 1090 |
| | | 327 |
| | | 33 |
| | | 3 |
| Gives the Content in *Wine Gallons* | | 123,49 |

*Or*, *By the Sliding-Rule*, I ſet 32,82 on D, to 40 on C, then againſt 32 and 26 on D, is 38,03 and 25,1 on C; therefore by the Rule, twice the firſt (38,03) *viz.* 76,06, added to 25,1 the ſecond, gives 101,16 *Ale Gallons*.

But 29,7 on D being ſet to 40 on C, then againſt 32 and 26 on D, is 46,4, and 30,7 on C; therefore twice (46,4) the firſt, *viz.* 92,8, added to 30,7, gives 123,5 *Wine Gallons*, both nearly the ſame as before.

PROP.

## PROP. II.

*The Length, Bung, and Head-Diameters of a Cask in the Shape of two equal Fruſtums of a* Circular Spindle *being given, to find its Meaſure in Gallons.*

*Solution.* To twice the Square of the Bung-Diameter, add the Square of the Head-Diameter, and let the Sum be called the *firſt Number*; then to the Square of the Length, add the Square of the Head-Diameter, and from the Sum take the Square of the Bung-Diameter, the Remainder multiplied by fourteen times the Square of the Difference of the Diameters is *a Dividend*: And thirty-five times the Square of the Length, added to five times the Square of the Difference of the Diameters, is the Diviſor: the Quotient of this Diviſion is the *Second Number*: Then if the Difference of the firſt and ſecond Numbers be multiplied by the Length of the Caſk, and that Product

by $\left\{\begin{matrix},0009284\\,0011333\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1077,15\\882,35\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{Ale,}\\\text{Wine,}\end{matrix}\right.$

the Product, or Quotient, will be the Meaſure required.

Or, By the Sliding-Rule,

Set $\left\{\begin{matrix}32,82\\29,7\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{Ale}\\\text{Wine}\end{matrix}\right\}$ on D, to the Caſk's Length on C, and find both the Bung and Head-Diameters, as alſo their Difference on D, noting the three Numbers oppoſite them on C; then if to the ſecond twice the firſt be added, and $\frac{3}{10}$ths of the third taken from the Sum, the Remainder will be *nearly* the Meaſure ſought.

*Example.* Let the Dimenſions of a Caſk ABCD (*Fig.* 59.) repreſenting two equal Fruſtums of a circular Spindle, be the ſame as in the preceding Propoſition.

OPERA-

OPERATION.

| | |
|---|---|
| By the laſt, the *firſt Number* is - | 2724 |
| The Square of (40) the Length is - | 1600 |
| The Square of (26) the Head-Diam. is | 676 |
| Their Sum - - - - - | 2276 |
| From which take the Square of (32) | 1024 |
| There remains - - - - - | 1252 |
| Which multiply by (14 x 6 x 6=) | 504 |
| | 5008 |
| | 6260 |
| The Product is the *Dividend*= - | 631008 |
| And 35 times the Square of 40 = | 56000 |
| To which 5 times 6 x 6 being added = | 180 |
| The Sum is the *Diviſor* - - - | 56180 |
| Then 56180)631008(= - - | 11,232 |
| Which taken from the 1ſt N°. leaves | 2712,768 |
| This multiplied by the Length - | 40 |
| Gives - - - - - - - | 108510,720 |
| Which multip. by ,0009284 invert. | 4829000,0 |
| | 97660 |
| | 2170 |
| | 868 |
| | 43 |
| Gives the Content in *Ale Gallons* | 100,741 |

*Or,*

| | |
|---|---|
| *Or*, - - - | 108510,720 |
| Multiplied by ,0011333 inverted | 33331100,0 |
| | 108511 |
| | 10851 |
| | 3255 |
| | 326 |
| | 33 |
| | 3 |
| Gives the Content in *Wine Gallons* | 122,979 |

*By the Sliding-Rule*, I ſet 32,82 on D to 40 on C, then againſt 32,26 and 6 on D, is 38,03, 25,1 and 1,34 on C; therefore 76,06 (=38,03 × 2) + 25,1—,4 (=1,34 ×, 3) gives 100,76 *Ale Gallons*.

And if 29,7 on D was ſet to 40 on C, then againſt 32,26 and 6 on D, is 46,42, 30,64 and 1,63 on C; therefore 92,84 (=46,42 × 2)+30,64 —,49 (=1,63 × 3) gives 122,99 *Wine Gallons*; both nearly the ſame as were found before by Computation.

## PROP. III.

*The Length, Bung, and Head-Diameters of a Cask* (Fig. 61.) *repreſenting two equal Fruſtums of a* Parabolic Spindle *being given, to find its Meaſure in Gallons.*

*Solution.* To twice the Square of the Bung-Diameter, add the Square of the Head Diameter, and from the Sum ſubſtract $\frac{2}{5}$ths of the Square of the Difference of the Diameters; then if the Remainder be multiplied by the Caſk's Length, and that Product

for

for { Ale / Wine } by { ,0009284 / ,0011333 } or divid. by { 1077,15 / 882,35

The Product, or Quotient, will be the Meaſure ſought.

*By the Sliding-Rule*, ſet { 32,82 / 29,7 } for { Ale / Wine } on D, to the Length on C, and find both the Bung and Head-Diameters, as alſo their Difference on D, noting the three Numbers oppoſite them on C; then if the ſecond is added to twice the firſt, and $\frac{2}{5}$ths of the third taken from the Sum, the Remainder will be the Meaſure of the Caſk propoſed.

For an *Example* to this Propoſition, let the Dimenſions of the Caſk be the ſame as thoſe before.

| | |
|---|---|
| Then twice the Square of the Bung added to that of the Head-Diam. } | = 2724 |
| $\frac{2}{5}$ths of (6x6) the Sq. of their Differ. is | 14,4 |
| Which taken from the firſt, leaves | 2709,6 |
| This multiplied by the Length, | 40 |
| Gives - - - | 108384,0 |
| And this mult. by ,0009284 invert. | 4829000,0 |
| | 97546 |
| | 2168 |
| | 867 |
| | 43 |
| Gives the Content in *Ale Gallons* = | 100,624 |

*Or*,

| | |
|---|---|
| *Or*, | 108384,000 |
| Multiplied by ,0011333 inverted | 33331100,0 |
| | 108384 |
| | 10838 |
| | 3252 |
| | 325 |
| | 33 |
| | 3 |
| Gives the Content in *Wine Gallons* = | 122,835 |

But, when 32,82 on D, is ſet to 40 on C, then againſt 32,26 and 6 on D, is 38,03; 25,1 and 1,34 on C: therefore $38,03 \times 2 + 51,1 - 1,34 \times ,4 = 100,62$ *Ale Gallons.*

And when 29,7 on D, is ſet to 40 on C, then on C againſt 32,26, and 6 on D is 46,42 30,64 and 1,63; hence $46,42 \times 2 + 30,64 - ,4 \times 1,63 = 122,83$ *Wine Gallons.*

## PROP. IV.

*The Length, Bung and Head-Diameters of a Cask,* (Fig. 62.) *repreſenting two equal Fruſtums of an equilateral* Hyperbolic Spindle *being given, to find its Meaſure in Gallons.*

*Solution.* Let twice the Square of the Bung-Diameter, added to that of the Head, be call'd the *firſt Number*; and to the Square of the Length add the Difference of the Squares of the Bung and Head-Diameters, let the Product of this by fourteen times the Square of the Difference of the Diameters, be *a Dividend*; alſo from 35 times the Square of the Length, take five times the Square of the Difference of the Diameters, the Remainder is *the Diviſor*; the Quotient of this Diviſion

is

is the *ſecond Number*. And the Difference betwixt the firſt and ſecond Numbers, multiplied by the Length of the Caſk, and that Product

for { Ale / Wine } by { ,0009284 / ,0011333 } { or divided by } { 1077,15 / 882,35 }

the Product, or Quotient, will be the Meaſure ſought.

*By the Sliding-Rule*; for { Ale / Wine } ſet { 32,82 / 29,7 } on D, to the Length on C, and find the Bung and Head-Diameters, as alſo their Difference on D, noting the three Numbers oppoſite them on C: Then if from the Sum of twice the firſt and ſecond $\frac{1}{2}$ the third be taken, the Remainder is the Meaſure requir'd.

*Example*. Let the Dimenſions of a Caſk repreſenting two equal Fruſtums of an equilateral hyperbolic Spindle be the ſame as before, and its Meaſure required in Ale, and Wine Gallons.

OPERATION.

| | |
|---|---|
| The *firſt Number* is - - | 2724 |
| The Square of (40) the Length = | 1600 |
| The Square of (32) the Bung-Diam. = | 1024 |
| Their Sum - - - - | 2624 |
| From which take the Square of 26 = | 676 |
| There Remains - - - | 1948 |
| This multiplied by 14×6×6 = | 504 |
| | 7792 |
| | 9740 |
| Gives *the Dividend* = | 981792 |

From

| | |
|---|---|
| From 35 times the Square of 40 = | 56000 |
| Take 5 times the Square of 6 = | 180 |
| The Remainder is the *Diviſor* = | 55820 |
| Therefore the 2*d Numb.* is $\frac{981792}{55820}$ = | 17,59 |
| Which taken from the 1ſt N°. gives | 2706,41 |
| This multiplied by the Length | 40 |
| Gives - - - - | 108256,40 |
| And this by ,0009284 inverted | 4829000,0 |
| | 97431 |
| | 2165 |
| | 866 |
| | 43 |
| Gives the Content in *Ale Gallons* | 100,505 |
| *Or*, | 108256,4 |
| Multiplied by ,0011333 inverted | 33331100,0 |
| | 108256 |
| | 10826 |
| | 3248 |
| | 325 |
| | 32 |
| | 3 |
| Gives the Content in *Wine Gallons* | 122,690 |

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But, *by the Sliding-Rule*, when 32,82 on D, is ſet to 40 on C; then againſt 32,26 and 6 on D, is 38,03, 25,1 and 1,34 on C; therefore $2 \times 38{,}03 + 25{,}1 - \frac{1}{2} \times 1{,}34 = 100{,}49$ *Ale Gallons*.

*Or*,

*Or*, 29,7 on D, ſet to 40 on C, againſt 32,26 and 6 on D is 46,42 30,7 and 1,63 on C, therefore $2 \times 46{,}42 + 30{,}7 - \frac{1}{2} \times 1{,}63 = 122{,}7$ *Wine Gallons*, both nearly the ſame as before.

## PROP. V.

*The Length, Bung, and Head-Diameters of a Cask,* (Fig. 63.) *in the Shape of two equal Fruſtums of a* Parabolic Conoid *being given, to find its Meaſure in Gallons.*

*Solution.* To the Square of the Bung-Diameter, add that of the Head, multiply the Sum by the Length of the Caſk, and this Product

for { Ale / Wine } by { ,0013925 / ,0017 } or divided by { 718,1 / 588,23 }

the Product, or Quotient, will be the Meaſure required.

*By the Sliding-Rule*: for { Ale / Wine } ſet { 26,8 / 24,25 } on D to the Caſk's Length on C, and find the Bung and Head Diameters on D, noting the two Numbers oppoſite them on C, whoſe Sum will be the Meaſure ſought.

*Thus in our Example*, The Square

{ 32 / 26 } the { Bung / Head } Diam. is { 1024 / 676 }

| | |
|---|---|
| Their Sum - - | 1700 |
| Multiplied by (40) the Length | 40 |
| Gives - - - - | 68000 |
| And this multiplied by .0013925 | 0,0013925 |
| | 111400000 |
| | 83550 |
| Gives the Content in *Ale Gallons* | 94,6900000 |

*Or*,

| | |
|---|---|
| Or, - - - | 68000 |
| Multiplied by - - | 0,0017 |
| | 476000 |
| | 68 |
| Gives the Content in *Wine Gallons* | 115,6000 |

By *the Sliding-Rule*, when 26,8 on D, is ſet to 40 on C, then againſt 32 and 26 on D, is 57,1 and 37,6 : whoſe Sum is 94,7 *Ale Gallons*.

But 24,25 on D, ſet to 40 on C, then againſt 32 and 26 on D, is 69,63 and 45,97 : whoſe Sum is 115,6 *Wine Gallons*, as before.

## PROP. VI.

*The Length, Bung, and Head-Diameters of a Cask* (Fig. 64.) *repreſenting two equal* Fruſtums *of an upright* Cone, *being given, to find the Meaſure in Gallons.*

*Solution.* Multiply the Sum of the Bung and Head-Diameters by the latter, and to the Product add the Square of the former; that Sum multiplied by the Length of the Caſk, and this

for { Ale / Wine } by { ,0009284 / ,0011333 } or divided by { 1077,15 / 882,35 }

the Product, or Quotient, will reſpectively be the Meaſure in Ale or Wine Gallons, as required.

By *the Sliding-Rule*; for { Ale / Wine } ſet { 65,64 / 59,4 } on D to the Caſk's Length on C; and on D find both the Sum, and Difference of the Diameters, noting the two Numbers oppoſite them on C; then if the ſecond is added to three times the firſt, four times the Sum is the Meaſure ſought.

*Example.* Let the Dimenſions of this Caſk be the ſame with the foregoing.

OPERATION.

| | | |
|---|---|---|
| The Bung } Diameter is { | - | 32 |
| The Head } | - | 26 |
| Their Sum | - - - | 58 |
| Multiplied by | - - | 26 |
| | | 348 |
| | | 116 |
| Gives | - - | 1508 |
| To which add the Square of 32 | = | 1024 |
| The Sum is | - - - | 2532 |
| This multiplied by the Length | | 40 |
| Gives | - - - | 101280 |
| And this by ,0009284 inverted, | | 4829000,0 |
| | | 91152 |
| | | 2026 |
| | | 810 |
| | | 41 |
| The Prod. is the Content in *Ale Gall.* | | 94,029 |
| *Or,* | - - - - | 101280 |
| Multiplied by ,0011333 inverted | | 33331100,0 |
| | | 101280 |
| | | 10128 |
| | | 3038 |
| | | 304 |
| | | 30 |
| | | 3 |
| Gives the Content in *Wine Gall.* | | 114,783 |

But,

But, by *the Sliding-Rule*, when 65,64 on D, is ſet to 40 on C; then againſt 58 (the Sum of the Diameters) and 6 (their Difference) on D, is 31,23 and ,34 on C reſpectively; therefore if to three times the firſt, *viz.* 93,69, the ſecond ,34 be added, we have 94,03 for the Meaſure in *Ale Gallons*.

*Or*, 59,4 on D ſet to 40 on C, then againſt 58 and 6 on D is 38,12, and ,41 on C; therefore, ,41 added to three times the firſt, *viz.* 114,36, gives 114,77 *Wine Gallons*, nearly the ſame as found by Computation.

We might have ſet $\left\{\begin{matrix}32,82\\29,7\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{Ale}\\\text{Wine}\end{matrix}\right\}$ on D, to the Caſk's Length on C; then if the Sum of the Diameters, and their Difference were found on D, three fourths of the firſt, added to one fourth of the ſecond, would have been the Meaſure of the Caſk propoſed.

## SCHOLIUM.

Thus we have done with the firſt thing propoſed in this Chapter, wherein, beſides the Theorems for computing the Meaſures of the ſeveral Varieties of Caſks mentioned at the beginning, we have given the Method of Gauging, each *by one Setting of the Lines* C,D *on the Sliding-Rule*, four of which are ſtrictly true, and the others (*viz.* thoſe for the *Circular* and *Hyperbolic Spindle*) ſo near, that they will never differ from the Truth, by the $\frac{1}{200}$th of a Gallon, in gauging any Caſk ever to be met with in Practice. And ſince the Operation is ſo very eaſy, I am ſurprized this Method was never practiſed; becauſe I am very certain, any of the abovementioned Caſks may be gauged with half the trouble required in the common Method of reducing them to Cylinders

with this Advantage, that here you are ever *ſure of the true Content*, if the Caſk has the aſſigned Form; whereas by any Rules hitherto publiſh'd for finding the *Mean Diameters*, it is doubtful.

And to render this Method ſtill more eaſy for Practice, I would adviſe every Officer that intends to make uſe of it, to have the Points anſwering the Numbers 32,28 and 29,7 on D, mark'd with a Braſs Pin, and A, W plac'd againſt them reſpectively, to denote Ale and Wine; which Points, in this way of Gauging, might be call'd *Gauge Points*, as 18,94 and 17,14 are in the common Method.

However, that this Treatiſe may not appear deficient, we ſhall now proceed to the ſecond thing propoſed, *viz.*

## SECTION II.

### *The Manner of Gauging a full Cask by the Mean Diameter.*

### PROP. I.

*The Bung and Head-Diameters of a Cask, reſembling the Fruſtum of a* Spheroid, *being given, to find the* Mean Diameter.

*Solution.* To the Head-Diameter add two thirds of the Difference of the Diameters, and to this add one third of the Quotient ariſing from dividing the Square of the Difference of the Diameters, by the Sum of the Head and twice the Bung-Diameters, ſo will you *nearly* have the Mean requir'd.

*Example.* Let the Bung be 32, and Head 26 Inches, (as in the laſt *Section*) then we have this

OPERATION.

OPERATION.

| | |
|---|---|
| The Head-Diameter is - | 26 |
| $\frac{2}{3}$ds of 6 the Diff. of the Diam. = | 4 |
| $\frac{1}{3}$ of $\frac{36}{99} = \frac{12}{90} = \frac{1,2}{90} =$ - | 0,1333 |
| Their Sum is the Mean, *viz.* | 30,1333 |

PROP. II.

*The Bung and Head-Diameters of a Cask, reſembling the Fruſtum of a* Circular Spindle, *being given, to find the Mean Diameter.*

*Solution.* To the Head-Diameter add two thirds of the Difference of the Diameters, and to this eleven ſixtieths ($\frac{11}{60}$) of the Square of the Difference of the Diameters, divided by the Head and twice the Bung-Diameter, the Sum will be the mean ſought, *nearly.*

Thus in this Example;

| | |
|---|---|
| The Head-Diameter is - - | 26 |
| $\frac{2}{3}$ of 6 - - | 4 |
| $\frac{11}{60}$ of $\frac{36}{99} = \frac{11 \times 6}{900} = \frac{,66}{9} =$ - | 0,0733 |
| Their Sum is the Mean, *viz.* | 30,0733 |

PROP. III.

*The Bung and Head-Diameters of a Cask, reſembling the Fruſtum of a* Parabolic Spindle, *given, to find the Mean Diameter.*

*Solution.* To the Head-Diameter add two thirds of the Difference of the Diameters, and to this four thirtieths ($\frac{4}{30} = \frac{4}{3}$) of the Quotient ariſing from dividing the Square of the Difference of the Diameters by the Head and twice the Bung-Diameter,

the Sum of theſe three Terms is the Mean ſought *nearly*.

| | |
|---|---|
| The Head-Diameter is - - | 26 |
| $\frac{2}{3}$ of 6 = - - | 4 |
| $\frac{4}{30} \times \frac{36}{90} = \frac{4 \times 12}{900} = \frac{48}{9} =$ | 0,0533 |
| Their Sum is the Mean, *viz.* | 30,0533 |

## Prop. IV.

*The Bung and Head-Diameters of a Caſk, repreſenting the Fruſtum of an* Equilateral Hyperbolic Spindle, *given, to find the Mean Diameter.*

*Solution.* To the Head-Diameter add two thirds of the Difference of the Diameters, and to this add one twelfth ($\frac{1}{12}$) of the Quotient found by dividing the Square of the Difference of the Diameters by the Head added to twice the Bung-Diameter, the Sum is nearly the Mean ſought.

| | |
|---|---|
| The Head-Diameter - - | 26 |
| $\frac{2}{3}$ of 6 = - - - | 4 |
| $\frac{1}{12}$ of $\frac{36}{90} = \frac{3}{90} = \frac{3}{9} =$ - | 0,0333 |
| The Sum is the Mean, *viz.* | 30,0333 |

## Prop. V.

*The Bung and Head-Diameters of a Caſk, in the Shape of the Fruſtum of a Parabolic Conoid, given, to find the Mean Diameter.*

*Solution.* To the Head-Diameter add half the Difference of the Diameters, and to this one fourth of the Square of the ſaid Difference, divided by the Sum of the Diameters, the Sum is the Mean ſought, *nearly*.

The

29) 4,5 (,1552
160
150
50

The Head-Diam. is 26
$\frac{1}{2}$ of 6 = - 3
$\frac{1}{4}$ of $\frac{36}{58} = \frac{1}{4} \times \frac{18}{29} = \frac{1}{2} \times \frac{9}{29} =$ 0,1552

The Sum is the Mean, *viz.* 29,1552

## PROP. VI.

*The Bung and Head-Diameters of a Caſk, reſembling the Fruſtum of an* Upright Cone, *given, to find the Mean Diameter.*

*Solution.* To the Head-Diameter add half the Difference of the Diameters, and to this one twelfth ($\frac{1}{12}$) of the Quotient found by dividing the Square of the ſaid Difference, by the Sum of the Diameters, the Sum is nearly the Mean requir'd.

58) 300 (,0517
100
420

The Head-Diam. is 26
$\frac{1}{2} \times 6 =$ - 3
$\frac{1}{12} \times \frac{36}{58} = \frac{3}{58}$ 0,0517

Their Sum is the Mean, *viz.* 29,0517

Having ſhewn how to find a Mean Diameter for any of the ſix Varieties, (*Page* 198.) that is to find the Diameter of a Cylinder of equal Length and Magnitude to a Caſk propos'd, which is deduc'd from the Theorems (*Page* 165.) we muſt now ſhew what Uſe is to be made of it when obtain'd: and as I ſuppoſe the chief Deſign of reducing Caſks to Cylinders, was from a Suppoſition that their Contents might thence be more eaſily computed by the Sliding-Rule, than when conſider'd in their original Forms, (tho' we have ſhewn it is otherwiſe) ſo here we ſhall only give the Method of finding the Content after that manner, the numeral Operation being eaſy from what was ſaid *Prop.* 1. *Chap.* IV. *Part.* 2.

## PROP. VII.

*The Length, Bung, and Head-Diameters of a Cask, repreſenting any of the ſix Varieties,* (Page 198.) *given, to find its Meaſure in Gallons by the Sliding-Rule.*

*Solution.* Firſt find the mean Diameter by one of the preceding Propoſitions, proper to the Caſk propoſed.

Then for { Ale / Wine } ſet { 18,94 / 17,14 } on D, to the Caſk's Length on C, and on C, oppoſite the mean Diameter on D, will be the Meaſure in Gallons of the Caſks propoſed.

*Example.* Let the Dimenſions of the ſeveral Varieties of Caſks be the ſame with thoſe in the preceding Propoſitions, we have ſhewn the Mean Diameters for them to be as hereunder, *viz.*

| | | | |
|---|---|---|---|
| For the Fruſtum of a | *Spheroid* | the Mean Diam. is | 30,1333 |
| | *Cir. Spin.* | | 30,0733 |
| | *Par. Spin.* | | 30,0533 |
| | *Equil. H. S.* | | 30,0333 |
| | *Par. Con.* | | 29,1552 |
| | *Cone* | | 29,0517 |

Therefore by the Rule above,

| | | | | |
|---|---|---|---|---|
| I ſet 18,94 on D, to 40 on C, | then againſt | 30,1333 | on D is | 101,2 |
| | | 30,0733 | | 100,7 |
| | | 30,0533 | | 100,6 |
| | | 30,0333 | | 100,5 |
| | | 29,1552 | | 94,7 |
| | | 29,0517 | | 94,0 |

respectively on C.

Which are the Contents in Ale Gallons of the above Caſks, and nearly the ſame as was found before in the laſt Section.

SCHO-

### Scholium.

The above Rules for finding the Mean Diameters are not only very near the Truth, but alſo very eaſy in Practice; however, they require a ſmall Multiplication and Diviſion, which is more than many care to be at the trouble of, rather chuſing to follow Methods leſs exact, that can be remembred, and ſo the Mean Diameter found as it were mentally: And therefore we muſt in this place ſet down ſome fixt Multipliers, which will ſerve pretty well, if (as commonly happens) the Bung-Diameter be about four or five times the Difference betwixt it and the Head.

They are thus for the ſeveral Varieties foregoing, *viz.*

| | | |
|---|---|---|
| For two Fruſtums of a | *Spheroid* | ,69 |
| | *Circular Spindle* | ,68 |
| | *Parab. Spindle* | ,67½ |
| | *Equil. Hyp. Spindle* | ,67 |
| | *Parab. Conoid* | ,52½ |
| | *Cone* | ,51 |

By which if the Difference betwixt the Bung and Head-Diameters of a Caſk reſembling any of the above Solids be multiplied, and the Product added to the Head-Diameter, the Sum will nearly give the Mean thereof.

The Methods deliver'd in this and the laſt Section would be ſufficient for Caſk-Gauging, as being each of them very exact, and (which indeed is chiefly to be conſider'd) ready in Practice, if there was not an Objection to both, which is too material to paſs over in ſilence: It is this; In each of them the *Shape* or Figure of the Caſk is ſuppoſed given, or eaſily determined by Inſpection; but this I conceive to be the greateſt Difficulty of all,

all, and is generally thought ſo by thoſe who have been moſt converſant therein: nay, if it was otherwiſe, why have ſo many Mathematicians, even thoſe of the firſt Rank, taken ſo much pains to give us Methods for determining the Loci deſcrib'd by Lines or Angles, moving according to ſome certain Laws? If it was ſo eaſy a matter to determine the Curve, (by whoſe Revolution on ſome Line as an Axis, any given Solid may be ſuppos'd generated) meerly by Inſpection? For in what I have juſt mention'd, you not only have an occular View of the Curve, but alſo the Law by which it is deſcrib'd; therefore we muſt have recourſe to ſome other Methods, which are built on a more certain Foundation, of which I think the following one (deduc'd from the laſt *Chapter* of *Part* 2.) is better fitted than any, or indeed the beſt the Nature of the thing admits of, whether we regard its Accuracy or Readineſs in Practice; but as therein a fourth Dimenſion is requir'd, *viz.* one more than in the other two, we muſt ſhew an eaſy Method of taking that at the ſame time the Length is meaſur'd; and this is the third thing propoſed in this Chapter.

## SECTION III.

### *Of Gauging a full Caſk by help of a* Fourth Dimenſion, *without knowing to which Variety it belongs.*

To render this Affair as clear as poſſible, let AA′DD′ (*Fig.* 66.) repreſent a Caſk, whoſe Head-Diameters are AD, A′D′, and Bung EF; alſo let the Center of each End be at O, O′: then if a ſtreight Ruler, having a Pin or ſomething of that kind in its Middle, which is put in the Bung-Hole at E, be turn'd about the ſame as a Center,

Center, till a Plumb-Line let fall from either end would paſs by O the Center thereof, and fixed in that Poſition; then two ſtreight Rulers being apply'd to each End, ſo as paſſing by O, O′, the Center of both, they would alſo paſs by the Ruler mention'd above, which we will denote by P P′, or *p p′*; 'tis plain from this Poſition of the Rulers if the Sum of the Parts AP, A′P′, or A*p*, A′*p*, which are eaſily meaſured, be taken from EF the Bung Diameter, you'll have AD, or A′D′ thoſe of the Head. And the Rulers remaining as before, if the Points M, M′, or *m*, *m′*, be taken in the middle betwixt the Points P, P′, or *p*, *p′*, and any common Foot-Ruler be apply'd ſucceſſively to M, M′, or *m*, *m′*, in a Poſition as nearly parallel to DP, DP, as poſſible, and ſlid down till it meet the Caſk, ſuppoſe in N, N′, you'll have the Meaſures of the Parts MN, M′N′, whoſe Sum being taken from EF, the Bung-Diameter, will leave NR, or N′R′, that in the Middle betwixt the Bung and Head: And laſtly, if the Ends of the Ruler PP′, which is fixt in the Bung, be moved in a vertical Poſition round the Pin at E (which is done by letting it conſtantly touch the Rulers DP, D′P′) till the Part A′P′ be equal to AP, then the Diſtance betwixt the Interſections of this Rule with thoſe applied to each End, (as before directed) will be the Length of the Caſk. And I muſt further obſerve, both this Length and conſequently the Middle-Points M, M′ would be obtain'd very readily, if the Rule PP′ was divided into Inches, and number'd both ways from E its Middle. But I ſhall not take upon me to give a particular Deſcription, how this Rule for meaſuring the Lengths of Caſks, and finding their middle Diameters, may be moſt conveniently made, that belonging to the Inſtrument-Makers; but I am perſuaded

ſuaded a better Contrivance than the Callipers might be found, which would render the following Method of Gauging vaſtly eaſy in Practice; and for its Truth, I will preſume to ſay, 'tis much more ſo, than any that has yet been offer'd for that purpoſe: for the Confirmation of which, I ſubmit to any one who will only take the trouble to gauge any Caſk by it, and after fill it with Water carefully meaſured.

### *The General* PROPOSITION.

*The Length, Bung, and Head-Diameters of a Cask, as alſo another Diameter taken in the Middle betwixt them, being given, to find its Meaſure in Gallons.*

*Solution.* To the Square of the Bung-Diameter add that of the Head, and four times the Square of the middle Diameter; let this Sum be multiplied by the Length of the Caſk, and that Product for

| | | | | |
|---|---|---|---|---|
| Ale | by | ,0004642 | or divided by | 2154,32 |
| Wine | | ,0005667 | | 1764,71 |

the Product, or Quotient, will be the Meaſure of the Caſk propoſed.

*By the Sliding-Rule*; For { Ale / Wine } ſet { 46,4 / 42, } on D, to the Caſk's Length on C, and find both the Bung, Head, and Middle Diameters on D, noting the three Numbers oppoſite them on C; then if to the Sum of the firſt and ſecond, four times the third be added, you'll have the Meaſure required.

*One Example will be ſufficient to illuſtrate this.*

Let the Length OO′ (*Fig.* 66.) of the Caſk AA′ DD′ be 40 Inches,

its { Bung / Head / Middle } Diameter { 32=EF / 26=AD / 30,4=NR } and its Meaſure required in Gallons;

| | | | |
|---|---|---|---|
| The Sq. of | 32 the Bung | Diam.= | 1024 |
| | 26 the Head | | 676 |
| 4 times the Sq. of | 30,4 the Mid. | | 3696,64 |
| Their Sum | | | 5396,64 |
| Multiplied by the Length = | | | 40 |
| | | | 215865,60 |
| And this by ,0004642 inverted | | | 2464000,0 |
| | | | 86346 |
| | | | 12952 |
| | | | 863 |
| | | | 43 |
| Gives the Content in Ale Gallons = | | | 100,204 |

Or, *By the Sliding-Rule*; I ſet 46,4 on D, to 40 on C, then againſt 32, 26, and 30,4 on D, is 12,8, 19 and 17, 1 on C, reſpectively; therefore 12,8+19+(4 x 17,1) 68,4=100,2 as found before by Computation; and after the ſame manner by uſing the Numbers proper for Wine Meaſure, the Content will be found 122,3 Wine-Gallons.

CHAP.

# CHAP. II.

## *Of* ULLAGING.

AS the *Poſitions* of a Caſk for the Convenience of drawing off, may with reſpect to its Axis (which is a Line ſuppos'd drawn from the Center of one End to that of the other) be either *upright* or *horizontal*, ſo from thence there will naturally be two Caſes in finding the Quantity of Liquor drawn out or remaining in a Caſk not quite full, and called *its Ullage*, which are very different from each other. To find the Ullage of an *horizontal* or *lying* Caſk, is a very difficult Problem, except for two of the Varieties, *viz.* the 5th and 6th; the firſt of theſe may be ſolved by *Prop.* 3. *Chap.* 4. *Part.* 2. and the ſecond from *Corol.* 5. *Prop.* 1. *Chap.* 5. of the ſame Part; alſo by the Theorem *Pag.* 147. you may nearly find the Ullage of any of the other Varieties; but they are all too tedious for Practice, ſo we muſt have recourſe to other Methods, of which that now generally followed, is by help of the Lines mark'd S L, S S, on the Sliding-Rule.

I cannot find for certain by whom theſe Lines were firſt introduced into Gauging, nor the Manner by which they are laid down; however, what I am going to offer on that Head, ſeems to be ſufficient for the Line S, L, which is by the help of the Table *Pag.* 116. *Edit.* 9. of *Everard's Steriometry*: For the Line of Numbers, or Slider N, (whoſe Conſtruction has been already ſhewn) being drawn out till 29.5 thereon (which is the Bung-Diameter of the Caſk, from whence this Experiment was made, and on which that Table is built)

coincide with a Point at the End of the Line SL mark'd 100, (which was oppoſite to 100 on N before it was drawn out) for then againſt the ſeveral dry Inches on the Line N, taken from his Table, muſt be the Points on the required Line SL, correſponding to the Number of Gallons of the Table that are reſpectively oppoſite the dry Inches, by which means thirty one Points of this Line will be had, and the remaining ones may eaſily be found by Proportion. But if this is not thought ſufficient, both it and the Line SS may be made after this manner: Let a Caſk be fixed upon, whoſe Form agrees the neareſt to the moſt common ones in Practice, and ſuppoſe both its Bung-Diameter and Length divided into 100 equal Parts, and Planes drawn through each of them, *viz.* the firſt hundred all parallel to the Heads, by which means this Caſk will be divided into a hundred Parts two ways, the firſt ſerving for the Line SL, and the ſecond for SS: For if by the preceding Propoſitions the Meaſure in Gallons of each of theſe Slices (which are parallel to the Axis) be ſucceſſively found, and the laſt always added to the Sum of the preceding ones, you'll have a Table of Meaſures analogous to that of Mr. *Everard's*; after which the Line S L may be made, as has been ſhewn above. In like manner, if the Meaſure in Gallons in each of the Slices parallel to the Heads of the Caſks are ſucceſſively found, and the laſt always added to the Sum of the preceding ones, a ſimilar Table will be had for a ſtanding Caſk, or for making the Line SS. And this, I imagine, was the Method by which they were actually made, becauſe the Line SL, at this time, does not correſpond every where with Mr. *Everard's* Table.

I thought it neceſſary juſt to hint at the Manner by which theſe Lines may be laid down, not ſo much for any Direction to the Inſtrument-Makers, for then I muſt have been more explicit, but that the Officer might be able to give ſome account (if required) of the Invention of every Line on a Rule he ſo frequently uſes. We ſhall now proceed to ſhew the manner of operating thereon.

## *The firſt Method of* ULLAGING.

### SECTION I.

*To find the Ullage of a ſtanding or lying Cask, by help of the Lines* SS *and* SL, *now on the Sliding-Rule.*

The General PROPOSITION.

*The Length, Head, and Bung-Diameters, as alſo the wet or dry Inches either in a ſtanding or lying Cask being given, to find the* Ullage.

SOLUTION.

Firſt by ſome of the preceding Propoſitions proper to the Caſk propoſed, find its Content; then for a {*Standing* / *Lying*} Caſk, ſet the {*Length.* / *Bung-Diam.*} on N to 100 on the Line {SS / SL} and againſt the wet or dry Inches on N, is the *reſerved Number* on {SS / SL}. Again; ſet 100 on B, to the whole Content on A, then againſt the reſerv'd Number on B is the Ullage on A, *viz.* what *remains* in the Caſk, if you uſed the *wet Inches*; but what is *drawn out*, if the *dry Inches*.

*Example.*

*Examp.* 1. Let the { *Length* / *Bung-Diam.* / *Head-Diam.* } of a ſtanding Caſk in the Shape of two Fruſtums of a parabolic Spindle be { 40 / 32 / 26 } Inches, and the wet Inches 28, to find the Ullage.

The whole Content of this Caſk, by *Prop.* 3. of the laſt *Chap.* is 100.6 Ale Gallons; and 40 the Length on N being ſet to 100 on SS, then againſt 28 the wet Inches, and 12 the dry is 71.6, and 28.4 the reſerved Numbers on SS reſpectively. Again, 100 on B being ſet to 100.6 the whole Content on A, then againſt 71.6 and 28.4, the reſerved Numbers on B, is 72 and 28.6 on A, whoſe Sum is 100.6 the whole Content; the *firſt* is what *remains* in the Caſk, and the *ſecond* what has been *drawn out*.

*Examp.* 2. Let the Caſk whoſe Dimenſions are the ſame with that above, be lying with its Axis parallel to the Horizon, and here alſo let the dry Inches be 12, and conſequently the wet Inches 20, to find the Ullage.

I ſet 32, the Bung-Diameter, on N to 100 on the Line SL, then againſt 12 and 20 on N is 32.8 and 67.2, the reſerved Numbers on SL. Again, 100 on B ſet to 100.6, the whole Content on A, then againſt 32.8 and 67.2, the reſerved Numbers on B, is 33 and 67.6 on A; the *firſt* is the empty part, or Liquor *drawn* out, and the *ſecond* what *remains*, whoſe Sum is 100.6, the whole Content. Theſe two Examples will be ſufficient to ſhew the Uſe of the Lines SS, SL, now put on the Sliding-Rule for Ullaging; but as they are laid down from ſome particular Caſks, ſo 'tis moſt certain they can only anſwer

for such as are similar to it, or nearly so; therefore in the third Method further on, I shall shew how others may be substituted in their place, which will render this difficult part of Gauging both readier in Practice, and more certain than heretofore.

## *The second Method of* ULLAGING.

## SECTION II.

### *To find the* Ullage *of a Cask by means of a Middle Diameter.*

### PROP. I.

*The Head-Diameter of an upright Cask, and that of the Liquor's Surface, as also another in the Middle betwixt it and the neareſt Head, being given, with the wet or dry Inches, to find the* Ullage.

### SOLUTION.

To the Square of the Head-Diameter, add that of the Liquor's Surface, and four times the Square of the Middle-Diameter, multiply the Sum by the wet or dry Inches (according to which is leaſt) and that Product for $\left\{\begin{matrix}Ale\\Wine\end{matrix}\right\}$ by $\left\{\begin{matrix},0004642\\,0005667\end{matrix}\right\}$ or divide it by $\left\{\begin{matrix}2154,32\\1764,71\end{matrix}\right\}$ the Product, or Quotient will be the Ullage ſought, *viz.* what is *drawn out* if the *dry* Inches are *leaſt*, what *remains* in if the *wet* Inches are *leaſt*, and *half* the Content of the Caſk when the *wet* Inches are *equal* to the *dry*.

*By the Sliding-Rule thus.* For $\left\{\begin{matrix}Ale\\Wine\end{matrix}\right\}$ ſet $\left\{\begin{matrix}46,41\\42,00\end{matrix}\right\}$ on D, to the Diſtance of the Liquor's Surface, and neareſt Head on C (that is, to the wet or dry

dry Inches on C) then on D find the Diameter of the Head, the Diameter of the Liquor's Surface, and twice the Middle Diameter, noting the three Numbers oppoſite them on C, whoſe Sum will be the Meaſure of the leaſt Fruſtrum or Ullage required.

*Example.* Let ABCD (*Fig.* 65.) be a ſtanding Caſk, whoſe Length OO′ is 40 *Inches*, Head-Diameter AD 26, that of the Liquor's Surface MM′ =31,04, the Diameter in the Middle betwixt them, *viz.* NN′=29,06, and OP the dry Inches 12, to find the Ullage.

OPERATION.

| | |
|---|---|
| The Square of 26 the Head-Diam. | 676 |
| The Square of 31,04 the Liquor's Diam. | 963,482 |
| The Square of 58,12 (=29.06 x 2) twice NN′= — — — | 3377,934 |
| Their Sum — — — | 5017,416 |
| Multiplied by the dry Inches — | 12 |
| Gives | 60208,992 |
| And this multiplied by ,0004642 (*inverted*) — — | 2464000,0 |
| | 24084 |
| | 3612 |
| | 241 |
| | 12 |
| | 27,949 |

Gives the Ullage or Meaſure of ADM′M in *Ale Gallons*; which is nearly one Gallon leſs than was found for this Caſk by the Line SS in the laſt Section, the Dimenſions above being ſuch as they would be found in a Caſk in the Shape of two equal Fruſtums of a parabolic *Spindle*. But if you would

would try the ſame by the Sliding-Rulet, ſet 46,41 on D, to 12 on C; then againſt 26,31 and 58,12 on D, is 3,76, 5,37, and 18,32 on C, whoſe Sum is 27,95 *Ale Gallons*, as found by Computation. The Dimenſions required in this Method of Ullaging may here eaſily be taken after this manner: The dry Inches OP being taken by dipping, let a ſtreight Ruler with a Plumb-Line at the end, be laid over the Center O, and moved backwards or forwards till the String juſt touch the Caſk as at E, then having made a Mark on the String at S and R, ſo that QS may be equal to the dry Inches, and QR its half; the Parts SM and RN are eaſily meaſured by applying a common Foot-Ruler parallel to QO thro' the Points R,S, and to touch the Staves at N, M; for then twice SM and twice RN taken from twice OQ (which is equal to the Bung-Diameter) will leave MM' and NN.

## Prop. II.

*The Length, Bung, and Head-Diameters of a lying Caſk, as alſo another in the Middle betwixt them, and the dry or wet Inches being given, to find the* Ullage *thereof.*

### Solution.

If the Caſk is *more* than half full, from the *dry Inches*, but if *otherwiſe*, from the *wet Inches*, ſubſtract half the Difference of the Bung and Head-Diameters, and alſo half the Difference of the Bung and Middle-Diameter, noting theſe two Remainders, let the leaſt be divided by the Head-Diameter, the biggeſt by the Middle-Diameter, and the wet or dry Inches (according to which is the leaſt) divided by the Bung-Diameter, look for the three Quotients in the Table (of the Areas of the

the Segment of a Circle) at the End under VS, taking thence the Number oppoſite them under the Letters *Seg.* multiply the leaſt of theſe by the Square of the Head-Diameter, the biggeſt by the Square of the Bung-Diameter, and the remaining one by four times the Square of the Middle-Diameter; and multiply the Sum of theſe Products by the Length of the Caſk, and that for $\left\{\begin{matrix}Ale\\Wine\end{matrix}\right\}$ by $\left\{\begin{matrix},000591\\,0007215\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1692\\1386\end{matrix}\right\}$ the Product, or Quotient will be the Ullage required, *viz.* what is *drawn out* if the wet Inches *exceed* the dry; what *remains in* when the dry inches *exceed* the wet, and *half* the Content of the Caſk when they are *equal*.

*Example.* Let AA′D′D (*Fig.* 66.) be a lying Caſk, the Length, Bung, and Head-Diameters 40, 32, and 26, as before; alſo let NR exactly in the Middle betwixt the Bung and Head-Diameters be 30.5, and the dry Inches, *viz.* EQ=12; to find the Ullage, or Meaſure of the empty Part EA′*o*′*o*A.

Half the Difference of the Bung and Head-Diameters is 3, half the Difference of the Bung and Middle-Diameters is ,75, both which being reſpectively taken from 12, the dry Inches, we have 9, 11,25: and 9, 11,25, and 12 being reſpectively divided by 26: 30,5 and 32 we have for Quotients ,346$\frac{1}{6}$. 368$\frac{5}{6}$ and 375 *nearly*; againſt which in the Table (when corrected for the fractional Parts) is ,241327—,263052—,269013.

OPERATION.

| | |
|---|---|
| The firſt multiplied by 26×26=676 gives | 163,137 |
| The ſecond multiplied by 61×61=3721 gives | 978,816 |
| The third multiplied by 32×32=1024 gives | 275,469 |
| Their Sum | 1417,422 |
| Multiplied by the Length | 40 |
| Gives | 56696,8800 |
| And this multiplied by ,000591 (*inverted*) | 195000,0 |
| | 283484 |
| | 51027 |
| | 567 |
| Gives the Ullage, or Meaſure of EA*oo*'A' in *Ale Gallons* | 33,5078 |

Which is half a Gallon more than was found for this Caſk by the Line SL.

*Example* 2. Let all the Dimenſions be the ſame as above, only here let the Middle Diameter be 29, ſo that the Caſk is conical; here the two Remainders are 9 and 10.5, and the three Quotients 26) 9 (,346$\frac{1}{6}$ : 29) 10.5 (.362$\frac{1}{14}$ : 32) 12 (,375; againſt which in the Table (after Correction) is ,241327 ,256539 ,269013.

OPE.

OPERATION.

| | |
|---|---|
| The firſt multiplied by 676 gives | 163,137 |
| The ſecond multiplied by 3364 gives | 862,998 |
| The third multiplied by 1024 gives | 275,469 |
| Their Sum is - - - | 1301,604 |
| Which multiplied by the Length | 40 |
| Gives - - - | 52064,1600 |
| And this multipl. by ,000591 *(inv.)* | 193000,0 |
| | 260321 |
| | 46858 |
| | 521 |
| Gives the Ullage in *Ale Gallons* | 30,7700 |

This Method is very exact, but too tedious for Practice, unleſs the Ullage be required to a great Nicety; however, it will ſerve to compare the former Method, and what I am going to explain in the next place.

## *The third Method of* ULLAGING.

## SECTION III.

*To find the* Ullage *of a Caſk by the Mean Diameter.*

### PROP. I.

*The Length of an upright Caſk, the Diſtance betwixt the Liquor's Surface, and its neareſt Head (that is, the wet or dry Inches) and a Diameter taken exactly in the Middle betwixt them, being given, to find the* Ullage.

SOLUTION.

Let the Square of the Middle Diameter be multiplied by the wet or dry Inches, *i. e.* by the leaſt of them, and this Product for { *Ale* / *Wine* } by { ,002785 / ,0034 } or divided by { 359,05 / 294,12 } the Product, or Quotient will be the Ullage required, if the Caſk is not half full, or very near it; but when that happens, the mean Diameter of the Caſk found by the laſt Chapter muſt be uſed inſtead of the ſaid Middle-Diameter.

The Reaſon of this Solution is from hence, that in an upright Caſk, not very near *half full*, the leſſer Segment may be conſidered as a Fruſtum of a parabolic Conoid, (becauſe the bulging Part is ſuppoſed to be wanting, which makes this Variety differ ſo much from the Spheroid and Spindles) and the Meaſure of every parabolic Conoid is equal to a Cylinder, whoſe Height is equal to that of the Conoid and Diameter of its Baſe exactly equal to that Diameter of the Conoid in the Middle betwixt its Ends,

*By the Sliding-Rule:* For { *Ale* / *Wine* } ſet { 18,94 / 17,14 } on D, to the Diſtance of the Liquor's Surface from the neareſt Head on C; then againſt the middle or mean Diameter on D, is the Ullage on C.

*Example.* Let the Length of an upright Caſk be 40, the dry Inches 12, and a Diameter exactly in the Middle betwixt the Head and Liquid Surface 29 Inches, to find the Ullage or Meaſure of the empty Part.

OPERATION.

| | |
|---|---|
| The Square of 29 the Mean Diam. is | 841 |
| Multiplied by the dry Inches - | 12 |
| Gives | 10092,000 |
| And this by ,002785 (*inverted*) | 587200,0 |
| | 20184 |
| | 7064 |
| | 807 |
| | 50 |
| Gives the Ullage in *Ale Gallons* - | 28,105 |

*Or by the Sliding-Rule :* Set 18.94 on D, to 12 on C; then againſt 29, the mean Diameter on D, is 28.1 on C, the Ullage required.

I muſt obſerve, this Method for the four firſt Varieties, gives the Ullage a ſmall matter too much, but is exact for the fifth, and a ſmall matter too little for the ſixth; but the Variation from Truth can never be material, but when the Caſk is near half full, and then the mean Diameter of the Caſk is to be taken for the Mean of the Fruſtum, to be meaſured as obſerved before; but theſe laſt Caſes are better performed by the following Propoſition.

## PROP. II.

*The Length, Bung, and Head-Diameters, as alſo the wet or dry Inches of a ſtanding Caſk, not full, being given, to find the Ullage.*

### SOLUTION I.

If the Curvature near the Bung be *great*, then multiply the Difference of the Squares of the Bung and

and Head-Diameters, by four times the Square of the Difference betwixt half the Length of the Caſk, and the wet or dry Inches; let this Product be divided by three times the Square of the Caſk's Length, and the Quotient taken from the Square of the Bung-Diameter; multiply the Remainder by the Difference betwixt the wet or dry Inches, and half the Length of the Caſk; then if this Product for $\left\{\begin{matrix}Ale\\Wine\end{matrix}\right\}$ be multiplied by $\left\{\begin{matrix},002785\\,0034\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}359,05\\294,12\end{matrix}\right\}$ the Product, or Quotient will ſhew, how much the Caſk is more or leſs than half full; and therefore, in the firſt Caſe, if it be taken from half the Content, you'll have the Quantity drawn out; but in the other Caſe, added to half the Content, what yet remains in.

2°. *But if the Curvature of the Caſk be not great*, multiply the Difference of the Bung and Head-Diameters, by four times the Square of the Difference betwixt the wet or dry Inches, and half the Length of the Caſk; let the Product be divided by the Square of the Caſk's Length, then this Quotient taken from the Bung-Diameter, will be the Diameter of the Liquor's Surface; which ſquared and added to twice the Square of the Bung-Diameter, let that Sum be multiplied by the Difference betwixt the wet or dry Inches, and half the Length of the Caſk, then this for $\left\{\begin{matrix}Ale\\Wine\end{matrix}\right\}$ multiplied by $\left\{\begin{matrix},0009284\\,0011333\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1077,12\\882,35\end{matrix}\right\}$ the Product, or Quotient will be, what the Caſk is more or leſs than half full, from whence the Ullage may be found as above.

The

The Reaſon of both theſe Solutions is, that that Part of an upright Caſk, not full, which lies betwixt the Liquor's Surface and Bung-Circle, may be conſidered as of the firſt Variety; and therefore meaſured by the Rule given for that purpoſe in the firſt Chapter, the Diameter of the Liquor in the firſt Solution being calculated for a Spheroid, and in the ſecond for a parabolic Spindle, as will eaſily be ſeen by conſidering what was ſaid *Corol.* 7, 8. *Chap.* 6. for finding the Middle-Diameter.

We ſhall illuſtrate both theſe Solutions in the *laſt Example*, where the Length, Bung, and Head-Diameters are 40, 32, and 26, the dry Inches being 12, ſo that the Difference thereof, from half the Length of the Caſk, is 8.

*By the firſt* SOLUTION.

| | |
|---|---|
| The Square of the B. Diam. is 32×32 = | 1024 |
| The Square of the H. Diam. is 26×26 = | 676 |
| Their Difference is - - - | 348 |
| Multiplied by 4×8×8 - - - | 256 |
| | 2088. |
| | 1740 |
| | 696 |
| Gives - - - - | 89088 |
| This divided by 3×40×40=4800 gives | 18,56 |
| Which taken from the Square of the Bung-Diameter, leaves - | 1005,44 |
| This multip. by 8, the Difference betwixt the dry Inches and half the Length, - - | 8 |
| Gives - - - - - | 8043,520 |

And

| | |
|---|---|
| *Brought over* | 8043,520 |
| And this multipl. by ,002785 (*inv.*) | 587200,0 |
| | 16087 |
| | 5630 |
| | 643 |
| | 40 |
| Gives what the Caſk is more than half full in *Ale Gallons* - | 22,400 |
| Which taken from half the Content | 50,3 |
| Leaves the empty Part = - - | 27,9 |

Nearly the ſame as found for the ſame in the preceeding Section.

*By the ſecond Method.*

| | |
|---|---|
| The Difference betwixt the Bung and Head-Diameter is - - | 6 |
| Four times the Square of 4x8x8 = | 256 |
| Their Product is - - - | 1536 |
| Which divided by the Square of the Length - - - | 1600 |
| Gives - - - - | ,96 |
| This taken from the Bung-Diameter, leaves that of the Liquor's Surface | 30,04 |
| Its Square is - - - | 963,482 |
| Twice the Square of 32 is - | 2048 |
| Their Sum is - - | 3011,482 |
| Multiplied by 20—12 = - | 8 |
| Gives | 24091,856 |

This

| | |
|---|---|
| *Brought over* | 24091,856 |
| This multipl. by ,0009284 (*invert.*) | 4829000,0 |
| | 21683 |
| | 482 |
| | 193 |
| | 10 |
| Gives alſo the Exceſs by which the Caſk is more than half full - | 22,368 |
| And this taken from half the Content | 50,3 |
| Leaves the Liquor drawn out - | 27,9320 |

Nearly the ſame with the true Solution.

So that from hence it appears, if the Caſk is upright and nearly half full, this Propoſition is to be uſed for finding the Ullage, but the preceding one when not near half full.

## PROP. III.

*The Length, Bung, and Head-Diameter of a lying Caſk, as alſo the wet or dry Inches being given, to find the Ullage.*

### SOLUTION.

By the laſt Chapter, find the mean Diameter proper to the Caſk propoſed, then to twice the wet or dry Inches, *viz.* the leaſt of the two, or either of them when equal, add the mean Diameter, and from the Sum take the Bung-Diameter; let half of this Remainder be divided by the mean Diameter, and the Quotient looked for under the Letters VS in the Table of the Segment of a Circle at the End, (whoſe Diameter is Unity) from whence take the Number againſt it under the Letters *Seg.* multiply this by the Square of the mean Diameter, and that Product by the Length of the Caſk, and this for { *Ale* / *Wine* } by { ,003546 / ,004329 }

or

or divided by $\left\{\begin{matrix}282\\231\end{matrix}\right\}$ the Product or Quotient will give the Ullage required, *viz.* what is *drawn out* if the dry Inches are *leaſt*; what *remains* if the wet Inches are *leaſt*, and half the Content of the Caſk when the wet Inches are *equal* to the dry.

*Example.* Let the Dimenſion of a lying Caſk of the third *Variety* be the ſame as at the Example of *Prop.* 2. *Sect.* 2. of this *Chap.* and the dry Inches alſo 12, to find the Meaſure of the empty Part.

Here the mean Diameter is 30,053 (See *Prop.* 3. *Sect.* 2. of the laſt *Chap.*) which added to 24, (twice the dry Inches) gives 54,053, and 32 the Bung taken thence, leaves 22,053, half of which is 11,0265; this divided by 30,053, the mean Diameter, gives ,367 *nearly*; againſt ,367 under VS is - - - - ,261284

This multiplied by the Square of 30,053 (*inverted*) - - - 81,309

2351556
7839
261
209

Gives - - - - - 235,9865
This multiplied by the Length - 40

Gives - - - - 9439,4600
And this multip. by ,003546 (*inv.*) 6453 00,0

283184
47197
3776
566

33,4723

*Example.*

*Example* 2. Let the Dimenſion and the dry Inches be the ſame as above, only here let the Caſk be of the *ſixth Variety*.

The mean Diameter here is 29,0522, which added to 24, gives 53,0522; from which 32 (the Bung-Diameter) being taken, leave 21,0522, the half of which 10,5261 being divided by 29,0522, gives 362$\frac{1}{3}$ *nearly*; againſt which in the Table of the Segments is - - - ,256792

This multiplied by the Square of 29,0522 (*inverted*) - - 30,448

205,4336
102717
10272
77

Gives - - - - - 216,7402
And this multiplied by the Length 40

Gives - - - 8669,6080
And this multipl. by 003546 (*inv.*) 6453000,0

260088
43348
3468
520

Gives the Ullage in *Ale Gallons* 30,7424

From what has been ſaid it appears, the firſt Example of this Section for a ſtanding Caſk, differs but ,$\frac{16}{100}$, the ſecond but $\frac{5}{100}$, and the third buut $\frac{16}{1000}$ths of a Gallon from the Truth, as found in the preceding Section; and the two laſt Examples for a lying Caſk are ſtill nearer, for they each of them vary but $\frac{3}{100}$ of a Gallon from what

what was ſhewn for the ſame Dimenſions in the ſecond Method by a Middle Diameter, and is much nearer than can be found by the Lines SS and SL on the Sliding-Rule: But as that Inſtrument is principally uſed in Practice, where the Fatigue of Computation is avoided as much as poſſible, I ſhall ſhew in what follows, how other Lines may be put on the Sliding-Rule, by which the laſt Method for Ullaging a lying Caſk will be rendred very ready in Practice; for it is known by Method 1ſt, now in uſe, there are no leſs than three Operations for finding the Ullage of a lying Caſk to be done on the Sliding-Rule; *Firſt* you muſt find the *whole Content*; *Secondly*, the *reſerved Numbers* by the Lines N and SL; and, *Thirdly*, the *Ullage* by the Lines A, B; whereas, in that I am going to offer, you'll have but two Operations. But I muſt firſt ſhew how theſe Lines are to be made.

## *The* Conſtruction *and* Uſe *of a* New Line *for* Ullaging a *Lying Caſk.*

In order to ſhew the Foundation of this Line, we muſt obſerve, that if S denote any Segment or Number in the Table at the End under the Letters *Seg.* and *d* be the Diameter of any Cylinder, then $d^2 \times S$ is the Area of a ſimilar Segment of its Baſe in Inches. Alſo if L denote the Length of the Cylinder, whoſe Diameter is *d*, then $\frac{d^2 \times S \times L}{282}$ is the Meaſure of a Slice thereof, parallel to the Axis in Ale Gallons, whoſe End is ſimilar to the Segment S; or $\frac{d^2 \times S \times L}{231}$ is the Meaſure of the ſame in *Wine Gallons*; call the Meaſure M: then

then firſt for Ale Gallons we have $\frac{d^2 \times S \times L}{282} = M$, or $\frac{282}{S} \times M = d^2 L$. Whence this Proportion on the Lines C, D on the Sliding-Rule, *viz.* $\frac{\sqrt{282}}{S}$ on D, is to L on C, ſo is *d* on D, to M on C. Whence 'tis apparent, if 282 was divided by every Number of our Table of Segments, made to 1000 verſed Sines, (inſtead of 500) and half the Logarithms of the Quotients laid on a Line ſuch as SL now is, on the Sliding-Rule, from the ſame Scale of equal Parts that the Line marked N was made from, thence might the Ullage of any lying Caſk be found without knowing the whole Content, as now practiſed; for as the laſt or leaſt Number on this new Line we are ſpeaking of for Ale Meaſure is $\frac{\sqrt{282}}{.785398} = 18.95$, ſo nothing further is requiſite after this Line is made, but to bring the mean Diameter of any Caſk on N, to 18.95 on this new Line (which would ſtand where 100 now is,) then againſt the Mean wet or dry Inches on N, would be areſerved Number on the new Line: And by ſetting this reſerved Number on D, to the Length on C, then againſt the *mean* Diameter of the Caſk on D, would be the Ullage on C, *viz.* what is drawn out, if you uſed the *mean dry Inches*, but what yet remains in, if you uſed the *mean wet Inches*: Where, by the *mean wet* or *dry Inches* is to be underſtood, what remains after half the Difference of the Bung and Mean Diameter is taken from the wet or dry Inches at the Bung. But I muſt be a little more explicit in this matter. It has already been obſerved, the firſt Number towards the right Hand of this Line will

be 18,95, and the Point anſwering to it, the very ſame with that anſwering to 100 now on the Line SL; it is alſo manifeſt the Numbers on our Line will increaſe towards the left Hand, or in a contrary Order to thoſe on the Line N, in ſuch manner, that the Number oppoſite 1, on the Line N, or againſt its Beginning, will be nearly 460,584. The ſame Obſervations ſerve alſo for the Line that is to ſhew Wine Meaſure, only that Number on this at the right Hand will be 17.14, and on the left Hand againſt 1 on N will be a Diviſion nearly anſwering to 416,87; but we may give a Rule for finding a Point on the Line N, againſt which muſt be the Diviſion on the Line we want to conſtruct, agreeing to any given Number.

*The Rule is this*, To make the Line for $\left\{\begin{matrix}\textit{Ale}\\\textit{Wine}\end{matrix}\right\}$ Meaſure divide $\left\{\begin{matrix}359,05355\\294,11824\end{matrix}\right\}$ by the Square of any given Number; look for the Quotient in Mr. *A. Sharp*'s Table of Segments, and note down the verſed Sine belonging to it; then againſt that verſed Sine on the Line N (not drawn out) is the Point of Diviſion on the new Line, belonging to the ſaid Number, whoſe Square was the Diviſor above.

*N. B.* I choſe to refer to *Sharp*'s Table of Segments, becauſe it is the largeſt, and moſt correct of any extant. But had this Table been made for a Circle whoſe Diameter (and not its Area) is Unity, then inſtead of the Numbers $\left\{\begin{matrix}359,05\ \&c.\\294,11\ \&c.\end{matrix}\right\}$ above, we muſt have taken $\left\{\begin{matrix}282\\231\end{matrix}\right\}$. I ſhall juſt give an Example of this Method, let the Dimenſions be exactly the ſame with thoſe in the laſt Example.

Ther

Therefore I ſet 29.05, the mean Diameter on N, to 18.94 on SA; then againſt 10.52 the mean dry Inches on N, ought to be 33.14 on the new Line AL the *reſerved Number*. Again, I ſet 33.14, the reſerved Number on D, to 40, the Length of the Caſk on C; then againſt 29.05, the mean Diameter on D, is 30.7 on C, which is the Ullage required, as was found by Computation.

Hence, from what has been ſaid, 'tis moſt evident the third Method of Ullaging is by far preferable to what is now practiſed, whether we regard Truth, or Facility of Operation; ſo that I do not queſtion but, ſome time or other, to find it generally practiſed.

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## CHAP. III.

### *Of Gauging Tuns, Backs, Coolers, &c.*

THE Veſſels called *Tuns* are of various ſorts, ſome having the Ends equal, others unequal but ſimilar, others unequal and diſſimilar, but in general their Sides are ſtreight, or ſuch, that a right Line applied from the top to the bottom, touches the ſame every where, and theſe are what I ſhall principally conſider in this place: for tho' in ſome Parts of *England* another ſort, where the Sides are not ſtreight, may occur, yet it is very ſeldom, and when they do, what will be found in the next *Chap. for Gauging Coppers and Stills*, will be ſufficient for them alſo, the Theorems for Meaſuring all kinds of ſtreight-ſided Tuns with paral-

lel Ends is laid down at *Corol.* 5. *Chap.* 6. *Pag.* 192, 193. ſo that all that remains in this Place is but to give the Solutions in Words.

In order to which, we muſt explain the Meaning of ſome Terms uſed herein.

*Definition* 1. When the Bottom of any Tun is parallel to the Horizon, or exactly level, it is called an *upright Tun*, but otherwiſe, an *inclined one.*

2. What Liquor is required juſt to cover every Part of the Bottom of an inclined Tun, is called the *Drip.*

Or if a Plane be ſuppoſed drawn through the higheſt Point of the Bottom of an inclined Tun, parallel to the Horizon, that part of the Tun contained between the bottom and horizontal Plane, is the *Drip* thereof.

3. That horizontal Plane that juſt touches the higheſt Point of the Bottom of an inclined Tun, is called the *horizontal Baſe*, and another that would juſt touch the loweſt Point of the top or other end, the *horizontal End.*

Thus, (*Fig.* 68.) A*a* being an inclined Tun, where A is the higheſt Point of the Bottom, *a* the loweſt Point of the Top, and AL, *al*, two horizontal Planes drawn thro' A*a*, then AL is the horizontal Baſe, and *a l* the horizontal End.

4. If from *l* any Point in the horizontal End a Perpendicular be let fall on the horizontal Baſe, *viz.* *l* P, it is called the *Height* of the Tun.

5. The Ends of Tuns are ſaid *to be parallely-poſited*, when the Sides (if Rectangles) or Axis (if Ellipſes) of one End, are parallel to thoſe of the other, *viz.* either when the longeſt Side, or Axis of the one End, is parallel to the longeſt or ſhorteſt Side of the other End.

6.

6. If a parallely-poſited Tun is cut thro' the middle of both its Ends by a Plane that is parallel to the Sides when they are Rectangles, or paſſing thro' the Axis when Ellipſes, the Section of this Plane, with the Ends, are called the *Lengths* or *Breadths* of both the Ends reſpectively; that is, if its Section with one End is called the *Breadth* thereof, then its Section with the other End is called the *Breadth* of that End alſo; and this tho' it ſhould be longer than the other Dimenſion of that End.

I was neceſſitated to introduce theſe Names for the ſake of Brevity, for it is common to call the longeſt Side of every Rectangle the *Length*, and the ſhorteſt the *Breadth*; but then we muſt have augmented the Number of Propoſitions in this Section, to prevent Confuſion; whereas from hence we can comprehend all the Caſes in one.

## PROP. I.

*The Lengths and Breadths of the Ends of an upright parallely-poſited Tun, as alſo its Height, being given, to find the Meaſure in Gallons.*

### SOLUTION.

To twice the Length of the Top, add the Length of the Bottom; multiply the Sum by the Breadth at the Top, and call the Product the *firſt Number*; alſo to twice the Length of the Bottom, add the Length of the Top; multiply the Sum by the Breadth at the Bottom, call the Product the *ſecond Number*; add theſe two Products together, and multiply the Sum by the Height of the Fruſtum, call that Product the *third Number*; then if the Ends are circular or Ellipſes for {*Ale* / *Wine*} multi-

multiply the third Number by $\left\{\begin{matrix},0004642\\,0005667\end{matrix}\right\}$ or divide it by $\left\{\begin{matrix}2154.3\\1764.7\end{matrix}\right\}$ the Product or Quotient will ſhew the Meaſure ſought.

But if the Ends are Rectangles, the third Number multiplied by $\left\{\begin{matrix},0005916\\,0007215\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1692\\1386\end{matrix}\right\}$ the Product or Quotient will be the Meaſure ſought in Ale or Wine, as above.

This Rule is general for meaſuring any upright parallely-poſited Tun, whether the Ends are Circles, Squares, Ellipſes, Rectangles, Equal or Unequal and ſimilarly poſited (as ſome expreſs themſelves) or not; I ſhall inſtance in two of the moſt difficult Caſes.

*Example the Firſt*; There is a Tun EF*gh* (*Fig.* 67.) woſe Ends are Rectangles,

The $\left\{\begin{matrix}\textit{Length}\\\textit{Breadth}\end{matrix}\right\}$ of the Top $\left\{\begin{matrix}ab=68\\cd=85\end{matrix}\right\}$

The $\left\{\begin{matrix}\textit{Length}\\\textit{Breadth}\end{matrix}\right\}$ of the Bottom $\left\{\begin{matrix}AB=100\\CD=80\end{matrix}\right\}$

and the Height 30 Inches, whoſe Content is required in Ale Gallons.

OPERATION.

| | |
|---|---|
| Twice the Length of the Top (68 x 2) | = 136 |
| The Length at the Bottom - - | = 100 |
| Their Sum - - - | = 236 |
| Multiplied by the Breadth at Top | = 85 |
| | 1180 |
| | 1888 |
| Gives the *firſt Number* - - | = 20060 |

Twice

*Brought over* = 20060

Twice the Length the Bottom - = 200
The Length at the Bottom - = 68

Their Sum - - - - 268
Multiplied by the Breadth at Bottom = 80

Gives the *ſecond Number* - = 21440

The Sum of the 1*ſt* and 2*d Numb.* = 41500
Mult. by the Height of the Tun = 30

Gives the *third Number* - - 1245000
This mult. by ,000591 (*invert.*) - 195000,0

622500
112050
1245

Gives the Meaſ. of the Tun in *Ale Gal.* = 735.795

In this Example, the longeſt Side of one End was put oppoſite the ſhorteſt Side of the other End, but in that which follows, I ſhall ſuppoſe the longeſt and ſhorteſt Side of one End reſpectively, oppoſite the longeſt and ſhorteſt of the other.

*Example* 2. Let A *b* (*Fig.* 67.) be an upright parallelly-poſited Tun, whoſe Ends are Ellipſes, its Height being 30 Inches as before; and

The { *Length* / *Breadth* } of the *Bottom* { AB = 148 / CD = 96 } *Inches*;

The { *Length* / *Breadth* } of the *Top* { *a b* = 92 / *c d* = 72 } *Inches*;

Twice the Length of the Top - = 184
The Length of the Bottom - = 148

Their Sum - - - - = 332

*Brought over* = 332
Multiplied by the Breadth of the Top = 72

664
2324

Gives the *firſt Number* - - = 23904

Twice the Length of the Bottom = 296
The Length of the Top - - = 92

Their Sum = 388
Mult. by the Breadth of the Bottom - 96

2328
3492

Gives the *ſecond Number* - = 37248

The Sum of the 1*ſt* and 2*d* Numb. = 61152
Multiplied by the Height - - = 30

Gives the *third Number* - 1834560, ...
This multip. by 000,4642 (*inv.*) = 2464000,0

733824
110074
7338
367

Gives the Content in *Ale Gallons* = 851,603

And from hence the Reader cannot be at a loſs to obtain the Meaſure of any of theſe Tuns when full, or even part empty, by taking the Dimenſions of the Tun at the Liquor's Surface.

But

But as it would be troubleſome always to go through with the Computation above, and which indeed, properly ſpeaking, is only fitted for a Tun that is full, it is common in Practice to have the Content of Veſſels of this kind for every Inch of their Depth, regiſtered at the Office; ſo that the Officer has no more to do, but to take the wet or dry Inches; and then from ſuch a Table he has the Liquor in any of them. Hence it is neceſſary to ſhew the Method of finding the Content of any Part of theſe Tuns, uſually called *inching* a Tun; which may be performed by the following Propoſition.

## PROP. II.

*The Lengths and Breadths of the Ends of any upright and parallely-poſited Tun, together with its Height being given, ti find the Content agreeing to any Number of wet or dry Inches.*

### SOLUTION.

Multiply the wet Inches by the Length of the Top, and the dry Inches by the Length of the Bottom, the Sum of theſe Products divided by the Height of the Tun, is the Length of the Liquor's Surface. Alſo, multiply the wet Inches by the Breadth of the Top, and the dry Inches by the Breadth at Bottom, the Sum of theſe Products divided by the whole Height of the Tun, is the Breadth of the Liquor's Surface. Thus having got the Length and Breadth of the Liquor's Surface, take thoſe of the Top or Bottom, and by the preceding Propoſition find the Content of the upper or lower Fruſtum; remembring if you uſe the Lengths and Breadths of the Top with thoſe of the Liquor's Surface, you muſt alſo uſe the dry Inches,

Inches, and the Reſult will ſhew what the Tun *wants* of being full; but if you uſe the Lengths and Breadths of the Bottom with thoſe of the Liquid's Surface, you muſt alſo uſe the wet Inches for the Height, and then the Reſult will be what Liquor remains in the Tun.

*Corol.* 1. Hence by ſuppoſing 1, 2, 3, 4, *&c.* for the dry Inches, and theſe ſubſtracted from the whole Height the wet Inches, you may by this Rule inch any ſtreight-ſided Tun, whoſe Ends are parallely-poſited, be they round, ſquare, elliptical, or rectangulat, ſimilar or diſſimilar.

*Corol.* 2. But from the Forms of theſe Tuns, it is evident the ſeveral Lengths are equidifferent, and ſo are the Breadths; ſo that one of each from the Top or Bottom being found, you'll from thence have their common Difference, which continually added to, or ſubſtracted from the preceding one, as the Caſe requires, will give the ſucceeding ones.

*Examp.* Let there be an upright parallely-poſited Tun, whoſe Ends are Ellipſes, the Length at $\left\{\begin{matrix} Top = 65 \\ Bott. = 110 \end{matrix}\right\}$ Inch. the Breadth at $\left\{\begin{matrix} Top = 60 \\ Bott. = 100 \end{matrix}\right\}$ Inches, and the whole Depth 12 Inches, to find the Length and Breadth at 1 Inch from the Top.

OPERATION.

| | |
|---|---|
| The wet Inches multiplied by the Length at Top, *viz.* - | $11 \times 65 = 715$ |
| The dry Inches multiplied by the Length at Bottom, *viz.* - | $1 \times 110 = 110$ |
| Their Sum is - - - | $= 825$ |
| Which divided by 12, the whole Height, gives - - | $68\frac{3}{4}$ |

There-

Therefore $68\frac{3}{4}-65=3\frac{3}{4}$ the common Difference of all the Lengths.

| | |
|---|---|
| Alſo the wet Inches mult. by the Breadth at Top, *viz.* | $11\times60=660$ |
| The dry Inches multiplied by the Breadth at Bottom, *viz.* | $1\times100=100$ |
| Their Sum is | $=760$ |
| Divided by 12, the whole Depth, gives | $=63\frac{1}{3}$ |

Therefore $63\frac{1}{3}-60=3\frac{1}{3}$ the common Difference of all the Breadths.

*Or*, If the Difference of the Lengths at Top and Bottom, *viz.* $110-65=45$, had been divided by 12, the whole Depth, the Quotient ($3\frac{3}{4}$) is the Difference of all the Lengths; and the Difference of the Breadth at Top and Bottom, *viz.* $100-60=40$ divided by 12, gives $3\frac{1}{3}$ for the Difference of all the Breadths, as found before. Therefore $3\frac{3}{4}$ added to 65, and to the Sum continually, gives all the Lengths, as in the 2*d* Column of the following Table; and $3\frac{1}{3}$ added to 60, and to the Sum continually, gives all the Breadths as in the 3*d* Column.

*Corol.* 3. We may alſo give a Rule for computing the Contents belonging to the ſeveral wet or dry Inches, from having three contiguous Contents given, which renders the inching of theſe ſtreight-ſided Tuns exceeding eaſy.

*Rule.* Let each of the three contiguous Contents be divided by its dry Inches, and the ſecond Quotient taken from the third; then if three times the Remainder be added to the firſt Quotient, and the Sum multiplied by the dry Inches belonging to the Content required, the Product will be that Content.

We will illuſtrate this in inching the Tun mention'd in the laſt Example; and in the firſt place

we muſt find (by *Prop.* 1.) the Contents belonging to the three firſt dry Inches, the Operation for them is thus, *viz.* for the

*1ſt.* $\overline{\overline{65\times 2+68\frac{3}{4}}\times 60+\overline{68\frac{3}{4}\times 2+65}\times 63\frac{1}{3}}\times \frac{1}{6\times 359{,}05} = 11.4884$ *Ale Gallons.*

*2d.* $\overline{\overline{65\times 2+72\frac{1}{2}}\times 60+\overline{72\frac{1}{2}\times 2+65}\times 66\frac{2}{3}}\times \frac{2}{6\times 359{,}05} = 24.2766$ *Ale Gall.*

*3d.* $\overline{\overline{65\times 2+76\frac{1}{4}}\times 60+\overline{76\frac{1}{4}\times 2+65}\times 70}\times \frac{3}{6\times 359{,}05} = 38.4340$ *Ale Gall.*

And theſe three Terms being found, will ſerve to find all the reſt without purſuing the above Operations any further; for call the firſt A, the ſecond B, the third C, and the reſt in order D, E, F, G, *&c.* then by the laſt Rule we have

$D=\overline{\frac{A}{1}+3\times\overline{\frac{C}{3}-\frac{B}{2}}}\times 4=54{,}0305$

$E=\overline{\frac{B}{2}+3\times\overline{\frac{D}{4}-\frac{C}{3}}}\times 5=71{,}1355$

$F=\overline{\frac{C}{3}+3\times\overline{\frac{E}{5}-\frac{D}{4}}}\times 6=89{,}8187$

*The Content at* 4, 5, 6 *Inches from the Top.*

And after this manner the Numbers in the 4*th* Column of the following Table were computed. Each of which taken from 238,9595, the whole Content of the Tun leaves thoſe in the 5th Column.

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*The*

*The* TABLATURE *for Inching an upright and paralelly-poſited Tun, whoſe Ends are Ellipſes.*

| Dry Inch. | Lengths | Breadth | Cont. from Top | Cont. from Bottom | Wet Inch. |
|---|---|---|---|---|---|
| 0 | 65 | 60 | 0 | 238,9595 | 12 |
| 1 | 68 3/4 | 63 1/3 | 11,4884 | 227,4711 | 11 |
| 2 | 72 1/2 | 66 2/3 | 24,2766 | 214,6829 | 10 |
| 3 | 76 1/4 | 70 | 38,4340 | 200,5255 | 9 |
| 4 | 80 | 73 1/3 | 54,0305 | 184,9290 | 8 |
| 5 | 83 3/4 | 76 2/3 | 71,1355 | 167,8240 | 7 |
| 6 | 87 1/2 | 80 | 89,8187 | 149,1408 | 6 |
| 7 | 91 1/4 | 83 1/3 | 110,1497 | 128,8098 | 5 |
| 8 | 95 | 86 2/3 | 132,1982 | 106,7613 | 4 |
| 9 | 98 3/4 | 90 | 156,0338 | 82,9257 | 3 |
| 10 | 102 1/2 | 93 1/3 | 181,7262 | 57,2333 | 2 |
| 11 | 106 1/4 | 96 2/3 | 209,3446 | 29,6147 | 1 |
| 12 | 110 | 100 | 238,9595 | 0 | 0 |

The Uſe of this Table is evident by Inſpection; for ſuppoſe we ſhould come to the above Tun, and find 7 wet, or 5 dry Inches, then to find what Liquor is taken out or yet remains in the ſame, 'tis but looking for 7 in the laſt Column, or 5 in the firſt, and oppoſite either in the 4th is 71,1355, what the Tun wants of being full, and in the 5th Column is 167,8240 what Liquor is in the ſame.

*Corol.* 4. But ſince Tuns generally are made to lean a little, and alſo have a certain Point fixt for taking the wet or dry Inches from, call'd the *Dipping-Place*, which is commonly ſo taken, as to be the eaſieſt come at; we muſt ſhew how theſe Tuns are to be inched, and a Table made thereof; in order to which, and for the ſake of Brevity, we

we ſhall refer to *Fig.* 68. where AL *a l* are the horizontal Ends of the inclin'd Tun A *a*, and their Diſtance, or the Height thereof, *l* P; then by the preceeding Method, the Part AL *a l* being inch'd and a Table made of the ſame, as has been ſhewn already, let the Meaſure of the *Hoof* or *Drip* ALB be found, by the general *Prop. pag.* 172. and its Corollaries; or otherwiſe, alſo from the aſſign'd Dipping-place, take the wet Inches belonging the Drip, then if this is added to all the wet Inches of AL *l a*, reckoning from the horizontal Baſe AL, you'll have the wet Inches for the inclin'd Tun, alſo the Meaſure of the Drip added to each of the Meaſures in the 5th Column will be the Contents of thoſe wet Inches. Then theſe ſeveral Contents taken from the whole Content of the Tun, contain'd betwixt the real Baſe AB, and Horizontal End *a l*, will leave the Contents of the dry Part, wanting the Hoof at Top; and the wet Inches taken from the whole Dip, from the Dipping-place to the Bottom of the Tun, will leave the Dry Inches correſponding.

To illuſtrate what has been ſaid, let *Fig.* 58. be a Tun whoſe horizontal Ends AL, *a l*, are the ſame with that we have inch'd above; its Height being *l* P=12 Inches; alſo let the Dipping-place be at D, and the whole Dip D*p*=15 Inches; of which ſuppoſe that Part betwixt *p* and the horizontal Bottom is 2 Inches, and the Meaſure of the Drip 15 Gallons; then by the above Method we have the following Table.

Dry

| Dry Inch. | Lengths | Breadths | Content from *a l* | Content from AB | Wet Inch. |
|---|---|---|---|---|---|
| 1 | 65 | 60 | 0 | 253,9595 | 14 |
| 2 | 68 3/4 | 63 1/3 | 11,4884 | 242,4711 | 13 |
| 3 | 72 1/2 | 66 2/3 | 24,2766 | 229,6829 | 12 |
| 4 | 76 1/4 | 70 | 38,4340 | 215,5255 | 11 |
| 5 | 80 | 73 1/3 | 54,0305 | 199,9290 | 10 |
| 6 | 83 3/4 | 76 2/3 | 71,1355 | 182,8240 | 9 |
| 7 | 87 1/2 | 80 | 89,8187 | 164,1408 | 8 |
| 8 | 91 1/4 | 83 1/3 | 110,1497 | 143,8098 | 7 |
| 9 | 95 | 86 2/3 | 132,1982 | 121,7613 | 6 |
| 10 | 98 3/4 | 90 | 156,0338 | 97,9257 | 5 |
| 11 | 102 1/2 | 93 1/3 | 181,7262 | 72,2333 | 4 |
| 12 | 106 1/4 | 96 2/3 | 209,3448 | 44,6147 | 3 |
| 13 | 110 | 100 | 238,9595 | 15 | 2 |

This Method ſeems to me more natural than that now practiſed, where ſomething is generally to be added or ſubſtracted, as is expreſſed in the Dipping-place: for here by entring the Table with any dry or wet Inches (taken at the Dipping-place) in the firſt or laſt Column, then in the *4th* and *5th* you have what is wanting to fill the Tun as it then ſtands, and what is then in the ſame, and this directly without being troubled to add or ſubſtract at all.

## SCHOLIUM I.

The Quantity of the Drip is commonly found in meaſuring as much Water, as will juſt cover the Bottom, and then alſo an Account may be taken of the Heights to which the Liquor riſes, in pouring in every Gallon, ſo as to be able to tell how much is in when the Bottom is not cover'd; for in all

all other caſes if the Depth or Dip is not an exact Number of Inches, you may find how much is to be added or ſubſtracted from the tabular Number by Proportion. Thus in the laſt Example, ſuppoſe we ſhould find at any time the wet Inches 7, 2, take the Content for 7 wet Inches from that for 8, and there remains 15,6050; then ſay if 1 gives 15,6050, ,2 will give 3,12100, and ſo much is to be added to 167,8240, which gives 170,945, *nearly* the Content belonging 7,2 wet Inches.

We have obſerved, if the Bottom of the Tun is even, and alſo the Tun it ſelf the Fruſtum of any Pyramid or Cone, the Quantity of the Drip may be computed by the Theorem, *Page* 174. and its Corollaries: but ſince the Inclination is commonly but ſmall, half the Meaſure of the Fruſtum ABML, (where ML is parallel to AB′) may be taken for the Meaſure of the Drip; for tho' that will always be a little different from Truth, unleſs in priſmatic or cylindric Tuns, or Coolers, yet the Error will not be of any Conſequence in Practice.

### Scholium II.

Backs, or Coolers, whoſe Sides are commonly at Right Angles to their Bottoms, which are always very broad in reſpect of their Height, that ſo the Liquor put into them, may the ſooner cool; I ſay theſe Veſſels are not only inch'd, but the Content found to every tenth of an Inch, which is called *tenthing* a Back; in order to which, the exact Depth of the Veſſel muſt be found, for the Bottoms being very broad, are very often uneven, and thence the Depths taken at different Places muſt be different, a ſmall Error in which would produce a conſiderable one in the Content: Therefore to find the true Depth, or

Dip

Dip of a Back, or Cooler, let any Number of Dips be taken, and all added together, then the *Sum* divided by the *Number* of Dips, will be the true Dip; after which you muſt try round the Veſſel, till you find a Place where the Dip is equal to that you before found for the mean Dip, which muſt be mark'd for the Dipping-place. But this may be aſſumed any where, for 'tis but marking how much the Dip taken there, differs from the mean Dip, and noting the Difference thus, let the mean Dip of any Back be 5 Inches, and by dipping at the aſſumed place, ſuppoſe we ſhould find 5,5 for the Dip there, thence 'tis evident, 5 muſt be taken from all the wet Dips, and it may be noted down thus D—,5; but had the Dip there been found 4,5, which is 5 leſs than the mean Dip, then the Dipping-place muſt be mark'd D+,5.

The Dipping-place and mean Dip being found, and mark'd as above, we ſhall next ſhew how the ſame is to be tenth'd; in the firſt place, let the Area of the Top or Bottom be found by ſome of the preceding Propoſitions, then this divided by 282, will ſhew how many Gallons the Back holds at 1 Inch depth, one tenth of which will be what it holds at $\frac{1}{10}$th of an Inch depth; this doubled, tripled, quadrupled, *&c.* will be the Contents at $\frac{1}{10}$th, $\frac{2}{10}$ths, $\frac{3}{10}$ths, $\frac{4}{10}$ths, *&c.* of an Inch of its Depth; and by repeating this Operation, 'till you have multiplied the Content at $\frac{1}{10}$th of an Inch in depth by the mean Dip, you'll obtain a Table of the Contents of the whole Back for every tenth of an Inch of its Depth, the laſt of which will be the whole Content of the Veſſel.

*For Example*, Let there be a Back reſembling a Parallelopipedon, whoſe Length is 140 Inches, Breadth 120 Inches, and let the Depths taken at

| |
|---|
| 5,1 |
| 4,8 |
| 5 |
| 5,2 |
| 4,6 |
| 4,9 |
| 4,5 |
| 4,7 |
| 5,2 |
| 4,9 |
| 48,9 |

ten different Places croſs-ways, be as in the Margin, where the Sum is 48,9, this divided by 10, the Number of Dips, or which is here the ſame thing, removing the decimal Point one place towards the left Hand, gives 4,89 for the mean Depth.

Next to tenth the ſame, becauſe the Baſe of this Back is a Rectangle 140 × 120 = 16800 is the Area thereof in Inches, which divided by 282, gives 59,57451, for the Content in Gallons; at 1 Inch depth therefore 5,95745 is the Content at $\frac{1}{10}$th of an Inch depth; and this reduced into Barrels,

*B. F. A. Gall.*

Firkins and Gallons, is 0, 0, 5,95745; 8 Gallons being 1 Firkin, and 4 Firkins 1 Barrel.

This ſet down in the following Manner, *viz.*

| *Depth* | *B.* | *F.* | *Content in Ale Gal.* |
|---|---|---|---|
| ,1 - | 0 : | 0 : | 5,95745 |
| ,2 - | 0 : | 1 : | 3,91490 |
| ,3 - | 0 : | 2 : | 1,87235 |
| ,4 - | 0 : | 2 : | 7,82980 |
| ,5 - | 0 : | 3 : | 5,78725 |
| ,6 - | 1 : | 0 : | 3,74470 |
| ,7 - | 1 : | 1 : | 1,70215 |
| ,8 - | 1 : | 1 : | 7,65960 |
| ,9 - | 1 : | 2 : | 5,61705 |
| 1,0 - | 1 : | 3 : | 3,57450 |
| 4,89 - | 9 : | 0 : | 3,31936 |

And multiplied by 2, 3, 4, 5, *&c.* to 10, gives the Contents of this Back for every tenth of an Inch in Depth, which is ſufficient for Uſe, ſo we have only to the tenthing for an Inch, added the

the whole Content, which is found by multiplying the laſt Number by the mean Depth 4,89.

And this will be ſufficient for Backs whoſe Ends are equal ſimilar Figures, and their Sides ſtreight; but ſhould they be otherwiſe, 'tis neceſſary to find the Content for every Inch of the Depth, by taking the Dimenſions in the Middle of every one of them, after which you may proceed as above, only the Operation, or the Number that is to be multiplied by 2, 3, 4, 5, 6, *&c.* will be different for every Inch of Depth, but the *Manner* is the ſame in all.

## *Of Gauging* MALT.

In making Malt they uſe a Ciſtern, Couch and Floor, each of which is generally in the Shape of a Parallelopipedon, ſo that if the Area of the Baſe of any of them in Inches be multiplied by the Depth of the Malt, (in the ſame Meaſure,) you will have the Quantity of Malt in cubic Inches, and this divided by 2150,42, or multiplied by ,0004651, the Quotient, or Product will be the Meaſure of the ſame in Buſhels. But we muſt obſerve that Malt being a dry Subſtance, does not like what is Liquid, form a Plane at the Top, parallel to the Horizon, ſo that 'tis the Officer's buſineſs to examine carefully if the Malſter has laid his Malt even; if not, he muſt conſider what is to be allow'd for the Irregularity: Now the Method to do that, is the ſame with what we gave before for taking the mean Depth of a Cooler, which is, by taking a ſufficient Number of Depths croſsways, and at each Angle of the Figure form'd thereby, for then the *Sum* of all the Depths divided by their *Number*, will give the *mean Depth*,

or ſuch a one, that the Area of the Baſe being multiplied thereby, and that by ,0004651, this laſt Product will give the Quantity of Malt in Buſhels.

How this may be effected by the Sliding-Rule, has been ſufficiently explained before, (*Pag.* 52, 53, 54, 55.) ſo that it would be needleſs to inſiſt any longer on it in this Place.

---

## CHAP. IV.

### *Of Gauging* Coppers, Stills, *and* Curve-ſided Tuns.

#### I. *To Gauge a Copper which is Broadeſt at Top.* See Fig. 69.

*Rule.* TAKE Croſs-Diameters, that is, two perpendicular to each other, at the Top (AB) at the Bottom (*q q*, where a Plane would juſt touch H, the Vertex of the Crown) and alſo others (at K) exactly in the middle betwixt the Top (AB) and Bottom (*q q*). Then add the Product of the Top-Diameters, the Product of the Bottom-Diameters, and four times the Product of the Middle-Diameters together, let this Sum be multiplied by (GH) *viz.* the Height of the Copper from its Top (G,) to the Vertex of the Crown (H,) and that by ,0004642, or divide it by 2114,32, the Product, or Quotient will be the Meaſure of the Copper, all but what is required to cover the Crown; and this muſt be found by meaſuring in Water: then this added to the former, will ſhew the Quantity of Liquor the ſaid Copper will hold.

II.

II. *To meaſure a Still, or Copper, when the Top is not the Broadeſt.* (Fig. 70.)

*Rule.* Suppoſe the ſame divided into two Fruſtums, by a Plane paſſing thro' the broadeſt Part (LL,) and let each of theſe Fruſtums be meaſured by the laſt Rule, then the Liquor requir'd to cover the Crown being added to the Sum of the Meaſures of the two Fruſtums will be the Meaſure of the Copper or Still propoſed.

This Solution is deduced from *Corol.* 4. to *Prop.* 6. *pag.* 191. But if the Veſſel is very large, you may divide the Diſtance betwixt the Top and moſt bulging Part, as alſo that betwixt the Bottom and moſt bulging Part into 2, 3, or more equal Parts, and take Croſs-Diameters at each Point, then find the Meaſure by uſing the Theorem ſtanding againſt 4 or 5 Ordinates, *Page* 187. but the Rules above will be found generally ſufficient for gauging any Still, Copper, or Curve-ſided Tun to be met with in Practice, therefore we ſhall proceed to illuſtrate them by Examples.

*Examp.* I. Let ABCD (*Fig.* 69.) be a Copper broadeſt at the Top, whoſe Height from the Top G, to the Vertex of the Crown H is 36 Inches, and let the Point K be taken exactly in the Middle betwixt G and H; and ſuppoſe the Croſs-Diameters at $\left\{\begin{matrix}H\\K\\G\end{matrix}\right\}$ be $\left\{\begin{matrix}110,0\\113,0\\116,0\end{matrix}\right\}$ and $\left\{\begin{matrix}111,0\\114,0\\115,5\end{matrix}\right\}$ then for this Copper we have the following Operation.

| | |
|---|---|
| The Rectangle of the Top-Diameters, *viz.* 116×115,5 = | 13398,0 |
| The Rectangle of the Bottom-Diamet. *viz.* 110×111 = | 12210,0 |
| 4 Times the Rectang. of the Mid. Diam. *viz.* 113×114×4 = | 51528,0 |
| Their Sum - - - | 77136,0 |

| | |
|---|---|
| *Brought over* | 77136,0 |
| Multiplied by the Height - - | 36 |
| | 462816 |
| | 231408 |
| Gives - - - - | 2776896 |
| And this by ,0004642 (*inv.*) - | 2464000,0 |
| | 11187584 |
| | 1666138 |
| | 111076 |
| | 5554 |
| Gives the Meaſure of the Part ABqq in *Ale Gallons* - | 1289,0352 |
| To which add the Liquor requir'd to cover the Crown - | 28, |
| And you'll have the Content of the whole Copper, *viz.* - | 1317,0352 |

*Examp.* II. Let ABCD (*Fig.* 70.) be a Tun where the greateſt Bulge is not at the Top AB, but lower down as at LL, and let the whole Height GH be 36 Inches, as before; of which the Diſtance of the Bulge LL, *viz.* GO is 15 Inches, and conſequently OH=21 Inches, alſo let the Points KK′ be taken in the middle betwixt the Points GO and OH, and Croſs-Diameter taken at the Points G, K, O, K′, H, which ſuppoſe as here under; *viz.* let the Croſs-Diameters

| | | | | |
|---|---|---|---|---|
| at G | be 80,5 | and | 80,8 | then for this Tun |
| K | 85,5 | | 86, | |
| O | 89 | | 89,5 | |
| K′ | 86,5 | | 87, | |
| H | 83 | | 83,5 | |

we have the following

OPE-

OPERATION.

| | |
|---|---|
| The Rectangle of the Top-Diameters, *viz.* 80,5 x 808 = | 6504,4 |
| The Rectan. of the greateſt Bulge-Diameters, *viz.* 89 x 89,5 = | 7965,5 |
| 4 Times the Rectang. of the Mid. Diam. at K, *viz.* 85,5x86x4= | 29412,0 |
| Their Sum - - - - | 43881,9 |
| Multiplied by the Height - | 15 |
| | 2194095 |
| | 438819 |
| | 658228,5 |
| And this by ,0004642 = = | 2464000,0 |
| | 26329 |
| | 3949 |
| | 263 |
| | 13 |
| Gives the Meaſure of Part ABLL in Ale Gallons - - | 305,54 |

Again,

| | |
|---|---|
| The Rectangle of the Bottom-Diameters, *viz.* 83x83,5 = - | 6930,5 |
| The Rectan. of the greateſt Bulge-Diameters, *viz.* 89 x 89,5 = | 7965,5 |
| 4 Times the Rectang. of the Mid. Diam. at K', *viz.* 86,5x87,x4= | 30102,0 |
| Their Sum = - - = | 44998 |

| | |
|---|---|
| *Brought over* | 44998 |
| Multiplied by the Height - - | 21 |
| | 44998 |
| | 89996 |
| | 944958,00 |
| And this by ,0004642 (*inverted*) | 2464000,0 |
| | 377983 |
| | 56697 |
| | 3780 |
| | 189 |
| Gives the Meaſure of Part CDLL in Ale Gallons - - | 438,649 |
| To which add the Meaſure of the Part ABLL - - | 305,54 |
| Then add the Liquor required to cover the Crown - - | 32 |
| And you'll have the Content of the whole Tun in Ale-Gallons - | 776,189 |

*Examp.* 3. Let ABCD (*Fig.* 71.) be a Still where the greateſt Bulge is at LL, and let the whole Height GH be 42,8 Inches, of which the Diſtance of the Bulge LL, *viz.* GO is 22,3 Inches, and conſequently OH=20,5 Inches; alſo let the Points K, K′, be taken in the Middle betwixt the Points GO, and OH, and Croſs-Diameters taken at the Points G, K, O, K′, H, which ſuppoſe as below, *viz.* let the Croſs Diameters

at { G, K, O, K′, H } be { 21, 45,4, 47,8, 47, 44 } and { 21, 46, 47,3, 47, 44 } then for this Still we have the following

Ope-

OPERATION.

| | |
|---|---|
| The Rectangle of the Top Diam. *viz.* 21 x 21 = - - - | 441 |
| The Rectan. of the Diameters of the greateſt Bulge, *viz.* 47,8 x 47,3 = | 2260,94 |
| 4 Times the Rectangle of the Mid. Diams. at K, *viz.* 45,4 x 46 x 4 = | 8353,6 |
| Their Sum - - - - | 11055,54 |
| Multiplied by the Height - | 22,3 |
| | 3316662 |
| | 2211108 |
| | 2211108 |
| Gives the Meaſure of Part ABLL * | 246538,542 |

Again,

| | |
|---|---|
| The Rectangle of the Bottom Diameters, *viz.* 43,5 x 44 = | 1914 |
| The Rectan. of the Diameters of the greateſt Bulge, *viz.* 47,8 x 47, 3 = | 2260,94 |
| 4 Times the Rectangle of the Mid. Diameters, *viz.* 47 x 47 x 4 = | 8836, |
| Their Sum - - - | 13010,94 |
| Multiplied by the Height - - | 20,5 |
| | 6505470 |
| | 2602188 |
| | 266724,270 |
| * | 246538,542 |
| | 513262,812 |

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| | |
|---|---|
| *Brought over* | 513262,812 |
| This multiplied by 0005666 (*inv.*) | 6665000,0 |
| | 256631 |
| | 30796 |
| | 3079 |
| | 308 |
| | 290,814 |
| Then add the Liquor required to cover the Crown - - | 7,5 |
| And you'll have the Content of the whole Still in Wine Gallons - | 298,314 |

The Rule made uſe of (in or about *London*) for gauging Stills, is to meaſure the globical Fruſtum, or Part, from the Nails to the Top, as by *Rule* 3. at *Page* 132. foregoing; and then the Body, or remaining Part, by taking Croſs-Diameters at the Middle of every 6, 8, or 10 Inches: from thence their correſponding Areas are found, and multiply'd into their reſpective Depths, will give as many Cylinders, as are the Numbers of Diameters taken; the Sum of which Cylinders, added to the globical Fruſtum above, and the Liquor required to cover the Crown, will give the Content of the Still required.

*As for Example*, Let ABCD (ſee *Fig.* 71.) be a Still, whoſe Depth GH is 42,8, and GK=7,3, conſequently KH will be 35,5, and their reſpective middle Croſs-Diameters as follows.

OPERATION.

| Inches | Diameters | Diameters | Areas | | Wine Gall. |
|---|---|---|---|---|---|
| 7,3 | { 21,0 - - 21,0 | } - | 1,50 | } 5,75 + 5,75 + 1,50 × $\frac{7,3}{3}$ = | 31,59 |
| | { 41,5 - - 40,8 | } - | 5,75 | | |
| 6,0 | - - 43,9 - - 43,2 | - - | 6,45 × 6 | = | 38,70 |
| 6,0 | - - 47,0 - - 46,2 | - - | 7,38 × 6 | = | 44,28 |
| 6,0 | - - 47,8 - - 47,3 | - - | 7,69 × 6 | = | 46,14 |
| 6,0 | - - 47,6 - - 47,4 | - - | 7,67 × 6 | = | 46,02 |
| 6,0 | - - 46,5 - - 46,5 | - - | 7,35 × 6 | = | 44,11 |
| 5,5 | - - 45 - - - 45,2 | - - | 6,91 × 5,5 | = | 38,03 |
| Depth 42,8 | | | To cover the Crown | | 7,50 |
| | | | Content | | 296,37 |

This Rule for the globular Part holds true in no caſe, except when the lower Diameter is the Diameter of the generating Circle, which was juſtly remarked to me ſome time ago, by my ingenious Friend Mr. *Geo. Powel*, who has had many Years

Years Experience in ſurveying the Diſtillery, and is well ſkill'd in theſe kind of Speculations, and has clearly demonſtrated, if four times the Square of the Altitude be not equal to the Difference of the Squares of the Diameters, (which he ſays he never yet found) the greateſt Diameter will not be that of the Sphere.

I ſhall here ſubjoin his Demonſtration, for the Satisfaction of thoſe chiefly, who are concern'd in that Branch of the Revenue, and have ſo much Curioſity as to know the true Reaſon of what ſo frequently occurs, and would not rely too much upon the Force of Cuſtom, and take things for granted upon an implicit Belief only, when there is to be had moſt evident and indubitable Principles from Geometry, to found a certain Theory upon.*

## SCHOLIUM I.

*To inch down curve-ſided Figures, as Stills, &c. repreſenting the Fruſtum of a Sphere, obſerve the following Articles.*

I. As eight times the Height of the propoſed Fruſtum, in Inches, is to the Sum of its Baſe and Top-Diameters, ſo is the Difference of thoſe Diameters to a fourth Number; from which Number increaſed by one Inch, take half the ſaid Height, and multiply the Remainder by ,0272, reſerving the Product.

II. Multiply the Square of half the Baſe, or greater Diameter, by ,0136, and from the Product increaſed by ,0090666, *&c.* take half the

* *Fig.* 7. *Plate* 2. Let AB = greater Diameter, EK the leſſer, and DC = Height, then $\frac{AB^2}{4} - \frac{EK^2}{4} = DC^2$, that is $AB^2 - EK^2 = 4DC^2$. *Q. E. O.*

reſerved

reſerved Product, the Remainder will be the Wine Gallons at 1 Inch deep, which alſo reſerve.

III. To the reſerved Product found by Article I. add ,0272, and to that Sum add alſo ,0272, and to this laſt Sum ,0272, and ſo on continually, till the Number of Additions, or Sums thus found, are equal to the Number of Inches in the Height of the Fruſtum leſſen'd by two.

IV. From the reſerved Remainder take the reſerved Product, and from the Remainder thus ariſing, take the firſt of the ſaid Sums (mention'd in Article III.) and from the next Remainder the ſecond of thoſe Sums, and from the next the third, and ſo on continually.

V. Add the Remainders, mentioned in the laſt Article, continually together, in order as they follow; the Sum of the two firſt will be the Wine Gallons contained at 2 Inches deep, of the three firſt at three Inches deep, &c.

*Example.* Let the Top-Diameter be 21, and the Baſe 41, and the Height of the Fruſtum 7 Inches.

Then it will be as 56 : 62 : : 20 : 22142, from which increaſed by 1, taking 3,5, there reſts 19,642; and this multiplied by ,0272, gives ,53421 for the reſerved Product: And therefore the Square of half the greater Diameter by ,0136 being 5,715400, we have 5,4573354, equal to the reſerved Remainder.

The Work will ſtand as follows;

| *Sums.* | *Remainders* | | *Inch.* | |
|---|---|---|---|---|
| ,53426 | 5,4573 | | 1 | 5,45733 |
| ,0272 | ,5343 | | | 4,92307 |
| ,56146 | 4,9230 | | 2 | 10,38040 |
| ,0272 | ,5614 | | | 4,36161 |
| ,58866 | 4,3616 | | 3 | 14,74201 |
| ,0272 | ,5886 | | | 3,77294 |
| ,61586 | 3,7730 | | 4 | 18,51495 |
| ,0272 | ,6158 | | | 3,15708 |
| ,64306 | 3,1572 | | 5 | 21,67203 |
| ,0272 | ,6430 | | | 2,51402 |
| ,67026 | 2,5142 | | 6 | 24,18605 |
| | ,6702 | | | 1,84376 |
| | 1,8440 | * | 7 | 26,0281 |

## SCHOLIUM II.

### *To take the Dimenſions of a Still.*

Firſt take its true Depth from the Center of the riſing Crown H, to the Center of the Collar g, by laying a flat Rule, or the like, acroſs the Middle thereof, which being done, you muſt deduct the Depth of the Collar from the Depth ſo taken; by which means you'll obtain the true Height GH. At the ſame time you may alſo find the Altitude of the globical Part or (Zone) ABEF, which is join'd to the Body of the Still EFCD with large Nails, ſo that the Place of Contact plainly appears by a Seam that goes quite round, and this is uſually done by extending a ſtring diametrically in the Seam EF, and by obſerving where

where it cuts your Dimenſion-Rod in K; if you take that Number from the whole Depth, it is evident you'll not only have the Height of the Zone KG, but alſo KH. In the next place, quarter the Still, and take Croſs-Diameters *d d*, *r r* in the Middle of every ſix, eight, or ten Inches, the latter is generally ſufficient and moſt expeditious.

If you make uſe of *Theorem Page* 191. at the greateſt Bulge, or broadeſt Part of the Still LL. At the Top AB, and the Bottom *q q*, where a Plane would juſt touch H, the Vertex of the Crown (by the help of your Dimenſion-Rod) take Croſs Diameters, alſo others, exactly in the Middle betwixt the greateſt Bulge LL, and each End, *viz.* AB and *q q*, as at KK′, which are all the Dimenſions required; after this manner the ſame may be obtain'd for Coppers and curve-ſided Veſſels.

## ADDENDA.

I. Tho' we have already given all the Rules for gauging Tuns and Caſks commonly practiſed, with ſome others that have not been publiſh'd before; yet I thought it would not be improper to add what follows, which in ſome Caſes may be of uſe.

The Theorems I am going to offer, are for gauging Caſks whoſe Heads are unequal, and conſequently unequally diſtant from the Bung or bulging Part, in which Claſs ſome or moſt Stills, Coppers, and Tuns with curved Sides may be rank'd.

Let ABCD (*Fig.* 72.) be a Veſſel made from ſome of the Varieties mention'd *Page* 198. by cutting off a Part O′P from one End, and put $\left\{\begin{matrix} AD=b \\ BC=H \\ EF=b \\ OO'=l \end{matrix}\right\}$

Then if the original Caſk was of the

1st, 3d, 5th, 6th } Variety of K =

$$\frac{l\sqrt{b^2-H^2}}{\sqrt{b^2-H^2}+\sqrt{b^2-b^2}} \quad \frac{l\sqrt{b-H}}{\sqrt{b-H}+\sqrt{b-b}} \quad \frac{l\times b^2-H^2}{2b^2-H^2-b^2} \quad \frac{l\times b-H}{2b-H-b}$$

} KO =

$$\frac{l\sqrt{b^2-b^2}}{\sqrt{b^2-H^2}+\sqrt{b^2-b^2}} \quad \frac{l\sqrt{b-H}}{\sqrt{b-H}+\sqrt{b-b}} \quad \frac{l\times b^2-b^2}{2b^2-H^2-b^2} \quad \frac{l\times b-b}{2b-H-b}$$

And having by theſe Theorems got the Diſtance of each Head from the bulging Part, the Content of both theſe Fruſtums will be found by *Chap.* 1. *Part.* 3. whoſe Sum will be the Meaſure of the Veſſel propoſed.

*Corol.* From theſe Theorems 'twill be eaſy to deduce Rules for computing a Diameter at a given Diſtance from the Bung-Circle for any of thoſe Varieties, and from thence the Reaſon of the Rule for Ullaging a ſtanding Caſk, *Page* 233.

3. *To find the true Content of the Globical Part of a Still in Wine Gallons.*

* *Rule.* To half the Sum of the Squares of the Top and Bottom Diams. add $\frac{1}{3}$ of the Square of the Altitude, multiply the Sum by the Depth, and this Product divided by 294, or multiplied by ,0034, and you'll have the Content in Wine Gallons.

*Examp.* Let the Top-Diameter be 21, Bottom-Diam. 41,2, and the Height 7,3 Inches.

OPERATION.

$21\times21=441$, $41,2\times41,2=1697,44$, and $7,3\times7,3\times\frac{1}{3}$

$$\therefore = 35,52^2\,\frac{441+1697,44}{2} + 35,52\times\frac{7,3}{294} = 27.43$$

Wine Gallons, *nearly.*

II. *The Method for computing Logarithms.*

We ſhall now give the Rules for computing the Tabular Logarithms, as promiſed *Page* 31. In order to which, let $a, b$ be any two Numbers, of which $b$ is the greateſt, put $a+b=s$, and $b-a=d$, alſo aſſume $n=,8685889638$ *&c.* then the Tabular Log. of $a, b$ are as follows, *viz.* L : *a*

$$L:a = L:b - \frac{n \times d}{s} - \frac{d^2}{3s^2}A - \frac{3d^2}{5s^2}B - \frac{5d^2}{7s^2}C \&c.$$

$$L:b = L:a + \frac{nd}{s} + \frac{d^2}{3s^2}A + \frac{3d^2}{5s^2}B + \frac{5d^2}{7s^2}C \&c.$$

Where A, B, C, *&c.* denote the Term going before reſpectively. But we may give an Approximation ſufficient for making the Logarithms to more Places than in moſt Tables now uſed. Thus

$$\left.\begin{aligned} L:a &= L:b - \frac{n \times 3ds}{3s^2 - d^2} \\ L:b &= L:a + \frac{n \times 3ds}{3s^2 - d^2} \end{aligned}\right\} \text{Or,}$$

$$\left.\begin{aligned} L:b &- \frac{nd}{s} \times \frac{15s^2 - 4d^2}{15s^2 - 9d^2} \\ L:a &+ \frac{nd}{s} \times \frac{15s^2 - 4d^2}{15s^2 - 9d^2} \end{aligned}\right\} \textit{nearly.}$$

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Theſe Approximations are but *Corollaries* to a general Method of converting a Series converging ever ſo ſlow into another converging faſter. We ſhall illuſtrate both by an *Example*, let the Logarithm of 11 be required, the Log. of 10 is 1, hence $s=21$, $d=1$; therefore by the Series this

OPERATION.

$$L:a = 1{,}00000000$$
$$\frac{nd}{s} = A = 0{,}04136138$$
$$\frac{d^2}{3s^2}A = B = \quad 3126$$
$$\frac{3d^2}{5s^2}B = C = \quad 5$$

$$L:a + A + B + C = 1{,}04139269 = \text{Log. } 11$$

*Or*, By the Approximation $\frac{15s^2-4d^2}{15s^2-9d^2} = \frac{6611}{6606}$, and $\frac{d}{s} = \frac{1}{21}$, therefore $\frac{,8685896 \times 6611}{6606 \times 21} = ,04139269$, to which add 1, the Log. of 10, then we have 1,04139269 for the Log. of 11, as found by the Series.

But if $a$ and $a+2$ are two even Numbers, then $a+1$ is an odd Number, put $y=2a^2+4a+1$, then the Log. of $a+1$ is expreſſed by this Series, *viz.*

$$\text{L}:a+1=\text{L}:\frac{a\times\overline{a+2}}{2}+\frac{n}{2}\times\frac{1}{y}+\frac{1}{3y^3}+\frac{1}{5y^5}+\frac{1}{7y^7},$$

*&c.* which converges ſo faſt, that 'tis ſcarce worth while to uſe any Arts to render it more eaſy in practice; however, ſome may be deſirous to have an Approximation, which is as follows, *viz.*

$$\text{L}:\overline{a+1}=\text{L}:\frac{a\times\overline{a+2}}{2}+n\times\frac{3y}{6y^2-2}$$ *nearly*; or (nearer

$$\text{L}:\overline{a+1}=\text{L}:\frac{a\times\overline{a+2}}{2}+\frac{n}{6y^2}\times\frac{15y^2-4}{5y-3}.$$

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A

## *A* TABLE *of the* Areas *of* Segments.

| *V. S.* | *Seg. Area.* | *V. S.* | *Seg. Area.* |
|---|---|---|---|
| 001 | 000042 | 034 | 008273 |
| 2 | 000119 | 35 | 008638 |
| 3 | 000219 | 36 | 009008 |
| 4 | 000337 | 37 | 009383 |
| 5 | 000470 | 38 | 009763 |
| 6 | 000618 | 39 | 010148 |
| 7 | 000779 | 40 | 010537 |
| 8 | 000951 | 41 | 010931 |
| 9 | 001135 | 42 | 011330 |
| 010 | 001329 | 43 | 011734 |
| 11 | 001533 | 44 | 012142 |
| 12 | 001746 | 45 | 012554 |
| 13 | 001968 | 46 | 012971 |
| 14 | 002199 | 47 | 013392 |
| 15 | 002438 | 48 | 013818 |
| 16 | 002685 | 49 | 014247 |
| 17 | 002940 | 50 | 014681 |
| 18 | 003202 | 51 | 015119 |
| 19 | 003471 | 52 | 015561 |
| 20 | 003748 | 53 | 016007 |
| 21 | 004031 | 54 | 016457 |
| 22 | 004322 | 55 | 016911 |
| 23 | 004618 | 56 | 017369 |
| 24 | 004921 | 57 | 017831 |
| 25 | 005230 | 58 | 018296 |
| 26 | 005546 | 59 | 018766 |
| 27 | 005867 | 60 | 019239 |
| 28 | 006194 | 61 | 019716 |
| 29 | 006527 | 62 | 020196 |
| 30 | 006865 | 63 | 020689 |
| 31 | 007209 | 64 | 021168 |
| 32 | 007558 | 65 | 021659 |
| 33 | 007913 | 66 | 022154 |

| *V. S.* | *Seg. Area* | *V. S.* | *Seg. Area.* |
|---|---|---|---|
| 067 | 022652 | 103 | 042687 |
| 68 | 023154 | 104 | 043296 |
| 69 | 023659 | 105 | 043908 |
| 70 | 024168 | 106 | 044522 |
| 71 | 024680 | 107 | 045139 |
| 72 | 025195 | 108 | 045759 |
| 73 | 025714 | 109 | 046381 |
| 74 | 026236 | 110 | 047005 |
| 75 | 026761 | 111 | 047632 |
| 76 | 027289 | 112 | 048262 |
| 77 | 027821 | 113 | 048894 |
| 78 | 028356 | 114 | 049528 |
| 79 | 028894 | 115 | 050165 |
| 80 | 029435 | 116 | 050804 |
| 81 | 029979 | 117 | 051446 |
| 82 | 030526 | 118 | 052092 |
| 83 | 031076 | 119 | 052736 |
| 84 | 031629 | 120 | 053385 |
| 85 | 032186 | 121 | 054036 |
| 86 | 032745 | 122 | 054689 |
| 87 | 033307 | 123 | 055345 |
| 88 | 033872 | 124 | 056003 |
| 89 | 034441 | 125 | 056663 |
| 90 | 035011 | 126 | 057326 |
| 91 | 035585 | 127 | 057991 |
| 92 | 036162 | 128 | 058658 |
| 93 | 036741 | 129 | 059327 |
| 94 | 037323 | 130 | 059999 |
| 95 | 038909 | 131 | 060672 |
| 96 | 038446 | 132 | 061348 |
| 97 | 039087 | 133 | 062026 |
| 98 | 039680 | 134 | 062707 |
| 99 | 040276 | 135 | 063389 |
| 100 | 040875 | 136 | 064074 |
| 101 | 041476 | 137 | 064760 |
| 102 | 042080 | 138 | 065449 |

| *V. S.* | *Seg. Area.* |
|---|---|
| 139 | 066140 |
| 140 | 066833 |
| 141 | 067528 |
| 142 | 068225 |
| 143 | 068924 |
| 144 | 069625 |
| 145 | 070328 |
| 146 | 071033 |
| 147 | 071741 |
| 148 | 072450 |
| 149 | 073161 |
| 150 | 073874 |
| 151 | 074589 |
| 152 | 075306 |
| 153 | 076026 |
| 154 | 076749 |
| 155 | 077469 |
| 156 | 078194 |
| 157 | 078921 |
| 158 | 079649 |
| 159 | 080380 |
| 160 | 081112 |
| 161 | 081846 |
| 162 | 082524 |
| 163 | 083320 |
| 164 | 084059 |
| 165 | 084801 |
| 166 | 085544 |
| 167 | 086289 |
| 168 | 087036 |
| 169 | 087785 |
| 170 | 088535 |
| 171 | 089287 |
| 172 | 090041 |
| 173 | 090797 |
| 174 | 091554 |
| 175 | 092313 |
| 176 | 093074 |
| 177 | 093836 |
| 178 | 094601 |
| 179 | 095366 |
| 180 | 096134 |
| 181 | 096903 |
| 182 | 097674 |
| 183 | 098447 |
| 184 | 099221 |
| 185 | 099990 |
| 186 | 100774 |
| 187 | 101553 |
| 188 | 102334 |
| 189 | 103116 |
| 190 | 103900 |
| 191 | 104685 |
| 192 | 105472 |
| 193 | 106261 |
| 194 | 107051 |
| 195 | 107842 |
| 196 | 108636 |
| 197 | 109430 |
| 198 | 110226 |
| 199 | 111024 |
| 200 | 111823 |
| 201 | 112624 |
| 202 | 113426 |
| 203 | 114230 |
| 204 | 115035 |
| 205 | 115842 |
| 206 | 116650 |
| 207 | 117460 |
| 208 | 118271 |
| 209 | 119000 |
| 210 | 119897 |

| V. S. | Seg. Area. | V. S. | Seg. Area. |
|---|---|---|---|
| 211 | 120712 | 247 | 150953 |
| 212 | 121529 | 248 | 151816 |
| 213 | 122347 | 249 | 152680 |
| 214 | 123167 | 250 | 153546 |
| 215 | 123988 | 251 | 154412 |
| 216 | 124810 | 252 | 155280 |
| 217 | 125634 | 253 | 156149 |
| 218 | 126459 | 254 | 157019 |
| 219 | 127285 | 255 | 157890 |
| 220 | 128113 | 256 | 158762 |
| 221 | 128942 | 257 | 159636 |
| 222 | 129773 | 258 | 160510 |
| 223 | 130605 | 259 | 161386 |
| 224 | 131438 | 260 | 162263 |
| 225 | 132272 | 261 | 163140 |
| 226 | 133108 | 262 | 164019 |
| 227 | 133945 | 263 | 164899 |
| 228 | 134784 | 264 | 165780 |
| 229 | 135624 | 265 | 166663 |
| 230 | 136465 | 266 | 167546 |
| 231 | 137307 | 267 | 168430 |
| 232 | 138150 | 268 | 169315 |
| 233 | 138995 | 269 | 170202 |
| 234 | 139841 | 270 | 171089 |
| 235 | 140688 | 271 | 171978 |
| 236 | 141537 | 272 | 172867 |
| 237 | 142387 | 273 | 173758 |
| 238 | 143238 | 274 | 174649 |
| 239 | 144099 | 275 | 175542 |
| 240 | 144944 | 276 | 176435 |
| 241 | 145799 | 277 | 177330 |
| 242 | 146655 | 278 | 178225 |
| 243 | 147512 | 279 | 179122 |
| 244 | 148371 | 280 | 180019 |
| 245 | 149230 | 281 | 180918 |
| 246 | 150091 | 282 | 181817 |

| V. S. | Seg. Area. |
|---|---|
| 283 | 182718 |
| 284 | 183619 |
| 285 | 184521 |
| 286 | 185425 |
| 287 | 186329 |
| 288 | 187234 |
| 289 | 188140 |
| 290 | 189047 |
| 291 | 189955 |
| 292 | 190864 |
| 293 | 191775 |
| 294 | 192684 |
| 295 | 193596 |
| 296 | 194509 |
| 297 | 195422 |
| 298 | 196337 |
| 299 | 197252 |
| 300 | 198168 |
| 301 | 199085 |
| 302 | 200021 |
| 303 | 201232 |
| 304 | 201841 |
| 305 | 202761 |
| 306 | 203681 |
| 307 | 204605 |
| 308 | 205527 |
| 309 | 206451 |
| 310 | 207376 |
| 311 | 208301 |
| 312 | 209227 |
| 313 | 210154 |
| 314 | 211082 |
| 315 | 212011 |
| 316 | 212940 |
| 317 | 213871 |
| 318 | 214802 |
| 319 | 215733 |
| 320 | 216666 |
| 321 | 217599 |
| 322 | 218533 |
| 323 | 219468 |
| 324 | 220404 |
| 325 | 221401 |
| 326 | 222277 |
| 327 | 223215 |
| 328 | 224154 |
| 329 | 225093 |
| 330 | 226033 |
| 331 | 226974 |
| 332 | 227915 |
| 333 | 228858 |
| 334 | 229801 |
| 335 | 230644 |
| 336 | 231389 |
| 337 | 232034 |
| 338 | 232579 |
| 339 | 233026 |
| 340 | 235473 |
| 341 | 236421 |
| 342 | 237369 |
| 343 | 238318 |
| 344 | 239268 |
| 345 | 240218 |
| 346 | 241169 |
| 347 | 242121 |
| 348 | 243074 |
| 349 | 244026 |
| 350 | 244980 |
| 351 | 245934 |
| 352 | 246889 |
| 353 | 247845 |
| 354 | 248801 |

| V. S. | Seg. Area. | V. S. | Seg. Area. |
|---|---|---|---|
| 355 | 249757 | 391 | 284568 |
| 356 | 250715 | 392 | 285544 |
| 357 | 251673 | 393 | 286521 |
| 358 | 252631 | 394 | 287498 |
| 359 | 253590 | 395 | 288476 |
| 360 | 254550 | 396 | 289453 |
| 361 | 255510 | 397 | 290432 |
| 362 | 256471 | 398 | 291411 |
| 363 | 257433 | 399 | 292390 |
| 364 | 258395 | 400 | 293369 |
| 365 | 259357 | 401 | 294349 |
| 366 | 260320 | 402 | 295330 |
| 367 | 261284 | 403 | 296311 |
| 368 | 262248 | 404 | 297292 |
| 369 | 263213 | 405 | 298273 |
| 370 | 264178 | 406 | 299255 |
| 371 | 265144 | 407 | 300238 |
| 372 | 266111 | 408 | 301220 |
| 373 | 267078 | 409 | 302203 |
| 374 | 268045 | 410 | 303187 |
| 375 | 269013 | 411 | 304171 |
| 376 | 269982 | 412 | 305155 |
| 377 | 270951 | 413 | 306138 |
| 378 | 271920 | 414 | 307125 |
| 379 | 272890 | 415 | 308110 |
| 380 | 273861 | 416 | 309095 |
| 381 | 274832 | 417 | 310081 |
| 382 | 275803 | 418 | 311068 |
| 383 | 276775 | 419 | 312054 |
| 384 | 277748 | 420 | 313041 |
| 385 | 278721 | 421 | 314029 |
| 386 | 279694 | 422 | 315016 |
| 387 | 280668 | 423 | 316004 |
| 388 | 281642 | 424 | 316992 |
| 389 | 282617 | 425 | 317981 |
| 390 | 283592 | 426 | 318970 |

| S. V. | Seg. Area. |
|---|---|
| 427 | 319959 |
| 428 | 320948 |
| 429 | 321938 |
| 430 | 322928 |
| 431 | 323918 |
| 432 | 324909 |
| 433 | 325900 |
| 434 | 326892 |
| 435 | 327882 |
| 436 | 328874 |
| 437 | 329866 |
| 438 | 330858 |
| 439 | 331850 |
| 440 | 332843 |
| 441 | 333836 |
| 442 | 334829 |
| 443 | 335822 |
| 444 | 336816 |
| 445 | 337810 |
| 446 | 338804 |
| 447 | 339798 |
| 448 | 340793 |
| 449 | 341787 |
| 450 | 342782 |
| 451 | 343777 |
| 452 | 344772 |
| 453 | 345768 |
| 454 | 346764 |
| 455 | 347759 |
| 456 | 348755 |
| 457 | 349752 |
| 458 | 350748 |
| 459 | 351745 |
| 460 | 352751 |
| 461 | 353768 |
| 462 | 354795 |
| 463 | 355732 |

| V. S. | Seg. Area. |
|---|---|
| 464 | 356730 |
| 465 | 357727 |
| 466 | 358725 |
| 467 | 359723 |
| 468 | 360721 |
| 469 | 361719 |
| 470 | 362717 |
| 471 | 363715 |
| 472 | 364713 |
| 473 | 365712 |
| 474 | 366710 |
| 475 | 367709 |
| 476 | 368708 |
| 477 | 369707 |
| 478 | 370706 |
| 479 | 371705 |
| 480 | 372704 |
| 481 | 373703 |
| 482 | 374702 |
| 483 | 375702 |
| 484 | 376702 |
| 485 | 377701 |
| 486 | 378701 |
| 487 | 379700 |
| 488 | 380700 |
| 489 | 381699 |
| 490 | 382699 |
| 491 | 383699 |
| 492 | 384699 |
| 493 | 385699 |
| 494 | 386699 |
| 495 | 387699 |
| 496 | 388699 |
| 497 | 389699 |
| 498 | 390699 |
| 499 | 391699 |
| 500 | 392699 |

It will be eaſy from what was ſaid above, for the Reader to apply theſe Theorems of himſelf.

*3dly.* Seeing Silver, Soap, Candles, Starch, and ſeveral other Commodities, are charged according to Weight, and ſome Frauds having been committed by uſing falſe Scales, the Officer ought to take particular Care in that Point, as to ſee if the Beam play equally free on both Sides the Fulcrum; he ought alſo to ſee if the Brachia are equal in Length, and the Beam equally heavy on both Sides; for any of theſe being different, will cauſe the Weight to vary from Truth; for which Purpoſe 'tis thought convenient to add the following

PROPOSITION.

*The Weights of any Body taken in each Scale of a Ballance of unequal Brachia's being given, to find its true Weight.*

SOLUTION.

Multiply the two Weights together, and extract the Square Root of the Product, that will be the Weight of the Body propoſed. *Fig.* 73.

For let the Weight of the Body be $x$, and when it is put in the Scale A, let it weigh $w$; but when put in the Scale B, let it weigh W; then from the known Law in *Mechan.* we have theſe Proportions, *viz.* DC : CE : : $w$ : $x$, and DC : CE : : $x$ : W, therefore $w : x :: x :$ W, or $x^2 = w \times$ W, that is $x = \sqrt{w \times W}$. *Q. E. O.*

*Corol.* Hence we may alſo find the Proportion of the Brachia's of theſe Scales, for DC : CE : : $w$ : $\sqrt{w \times w}$.

N. B.

N. B. *'Tis thought proper to add the following Remark to* Scholium 2. Pag. 256.

The Bottoms of Common Brewers large Backs do generally warp after they have been a little time uſed, and become more and more uneven as they grow older, eſpecially ſuch as are not every where well and equally ſupported; many of them are ſo large and uneven, that 'tis hardly poſſible to find a true Medium of the Depth of the Wort, without taking a very great Number of Depths, and ſome are ſo ſituate, that it is difficult to take Depths in every Part where it may be neceſſary, and therefore it is a common Practice to take one Depth at a Place in the Back (cloſe to the Side) which may be always moſt conveniently come at; then ſee all the Wort let down into a Tun, in which it may be exactly gauged, and (if the Dip taken in the Back does not happen to be a mean one) mark on the Side of the Back ſuch Addition or Abatement as will make the Gauge of the Wort in the Back equal to the Gauge thereof in the Tun, which is called ſetting a Back by ſeeing the Wort come down; and this is frequently done, becauſe moſt large Backs are continually ſettling more one way than another. Beſides,

The Dimenſions of large Backs are (for want of more proper Inſtruments) generally taken with inched Tapes, which are ſeldom very exactly divided, and the Alteration of Weather affects them in their Lengths; and a very ſmall Error in the Dimenſions of ſuch a large Veſſel, cauſes a conſiderable one in its Area: but by ſetting the Back as above, there will be a Compenſation in the Depth for any Error that may happen in taking the other Dimenſions. So we ſhall here put a Period to this Treatiſe.

*FINIS.*

3. 11. 08.

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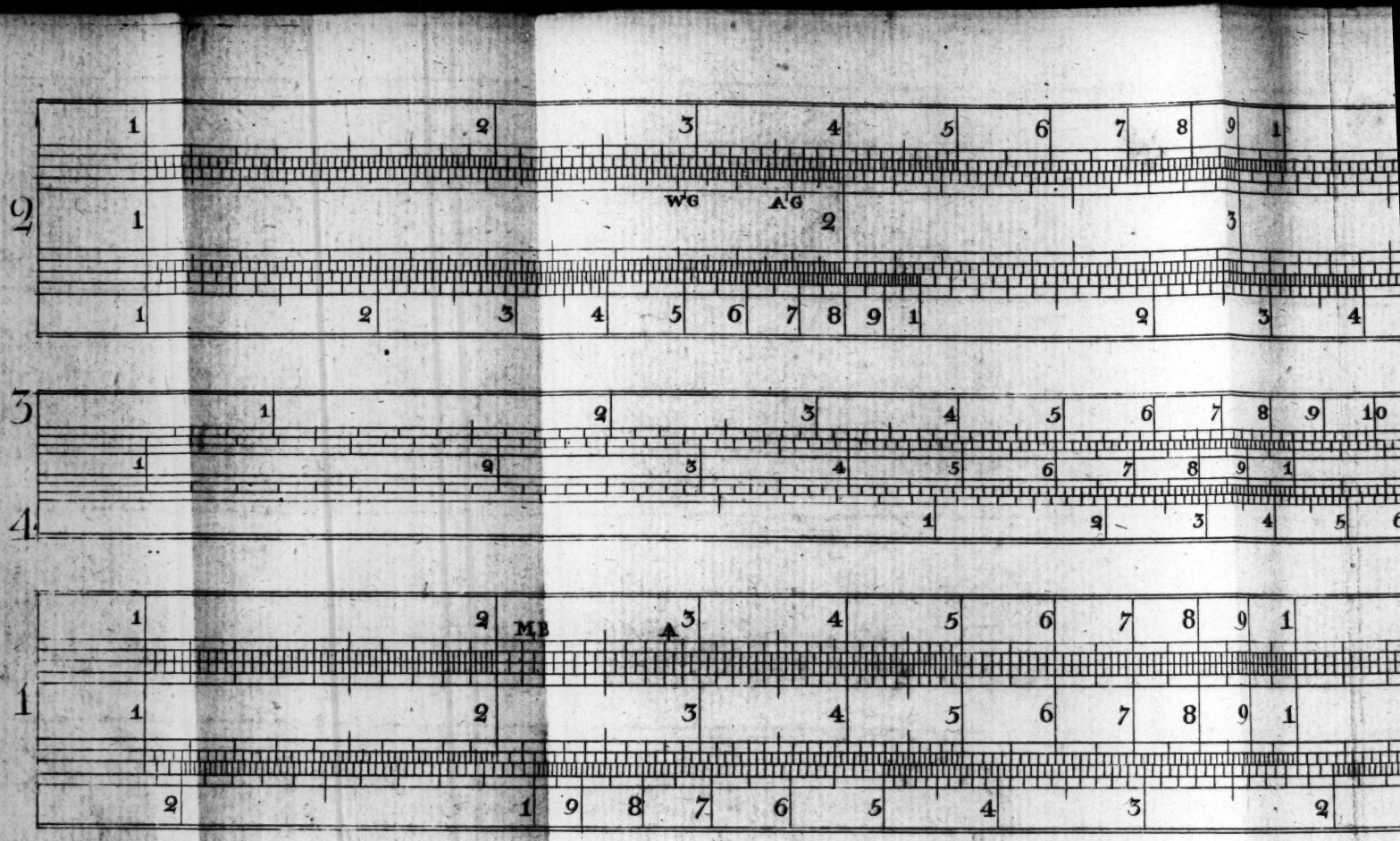
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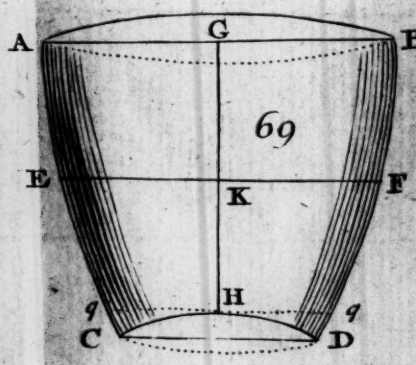
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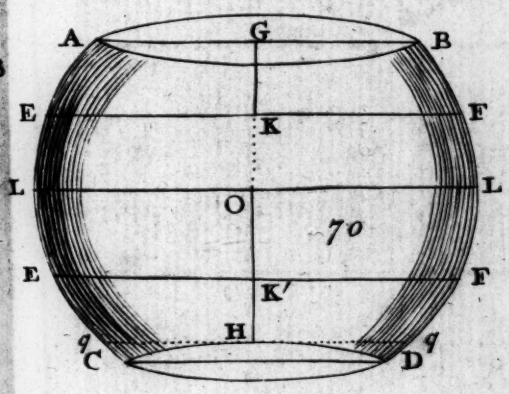
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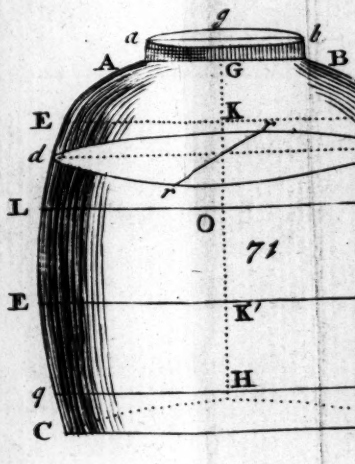
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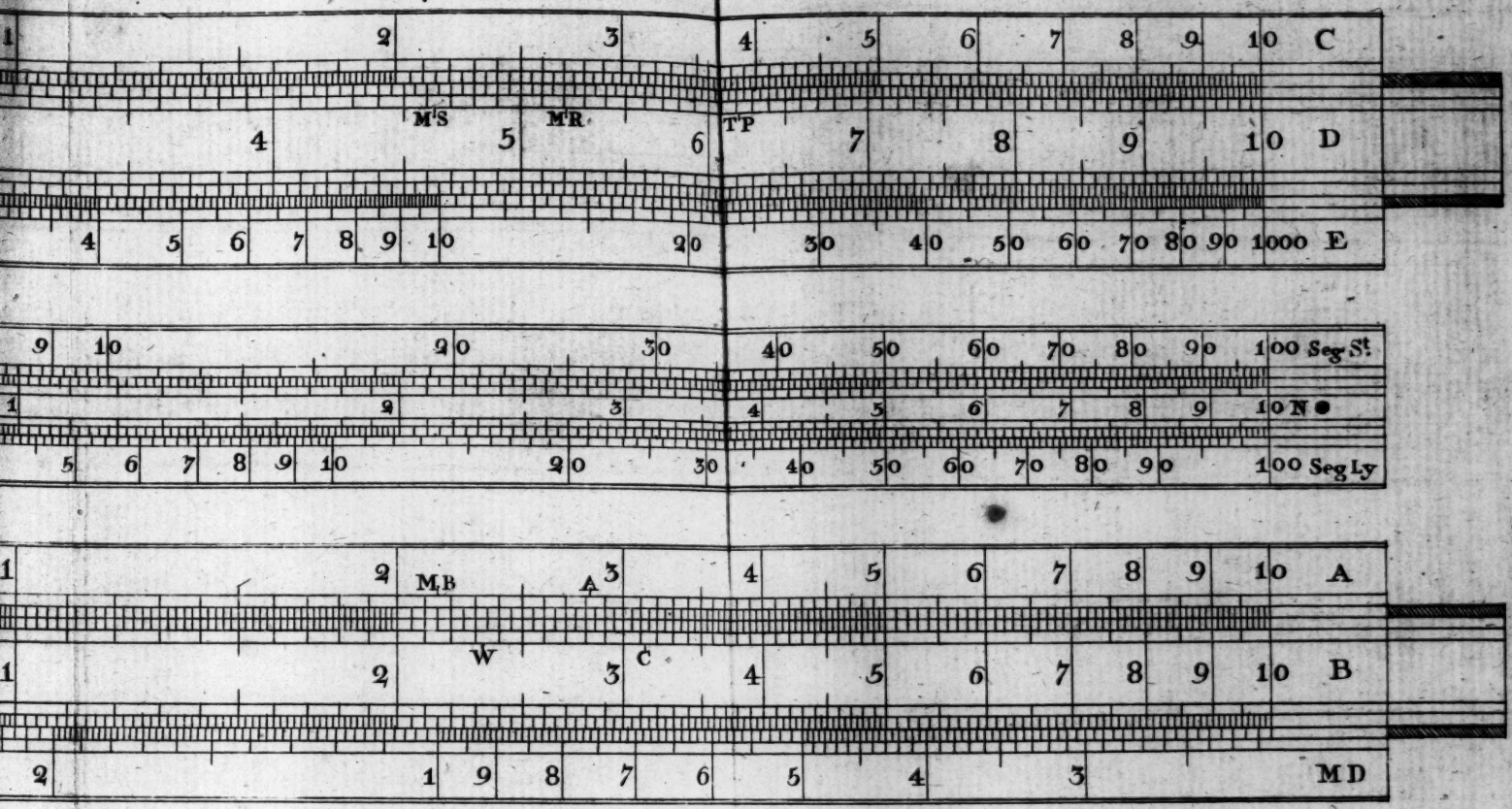
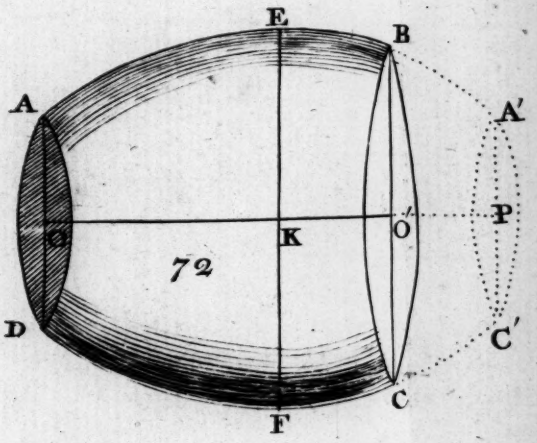
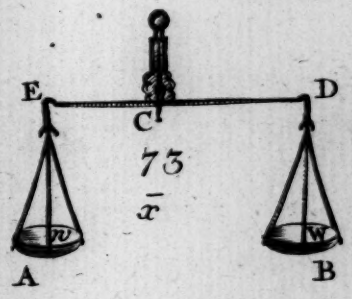
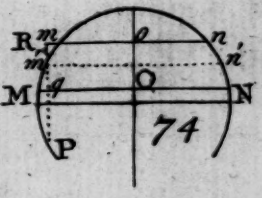
# Plate I



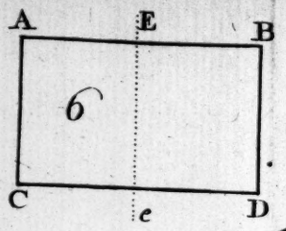


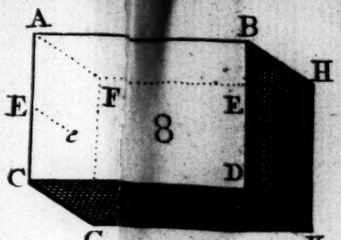



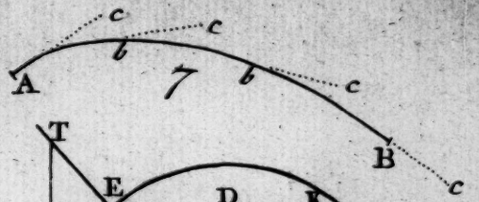



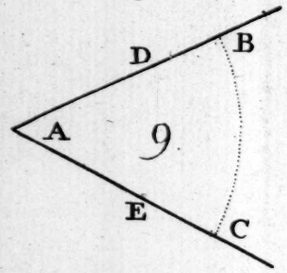

Will: Wyeth *Sculp.* 1740.

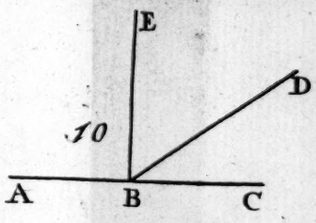
Plate. II.

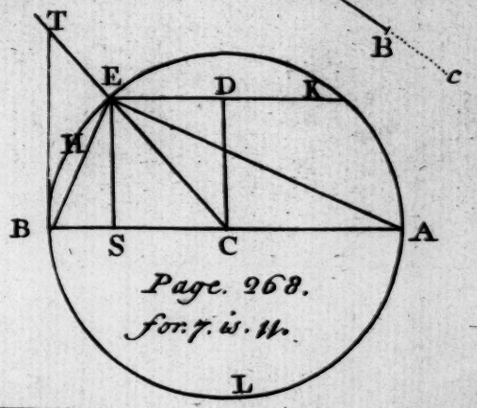


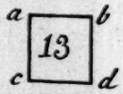



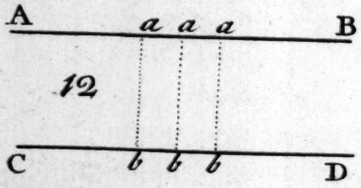



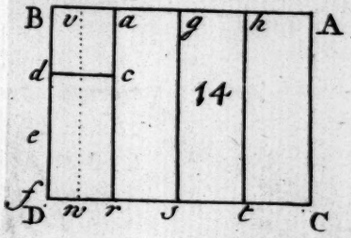



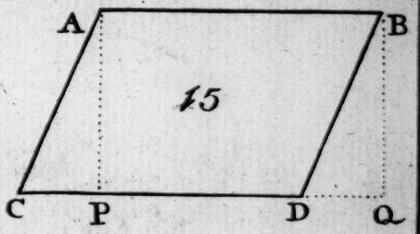



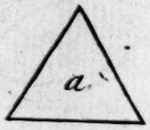



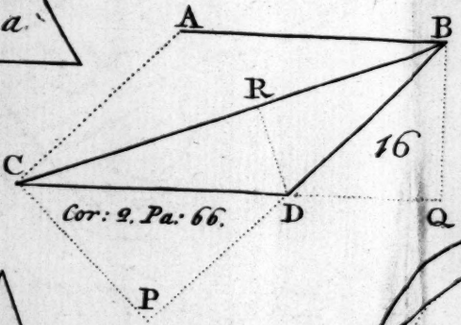



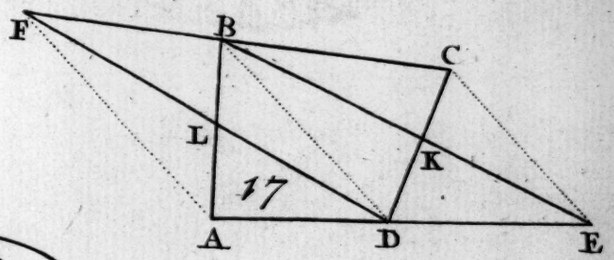



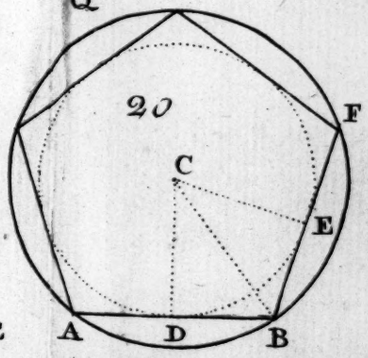



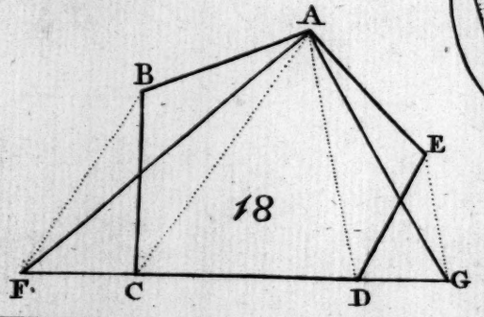



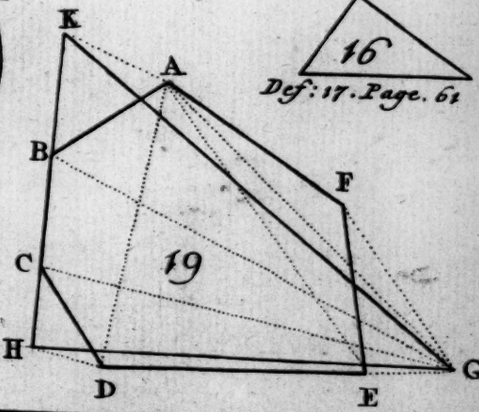



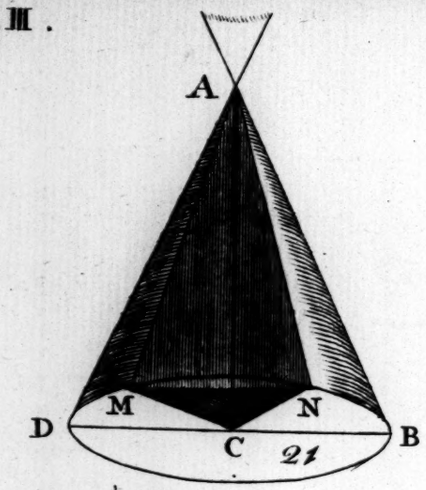


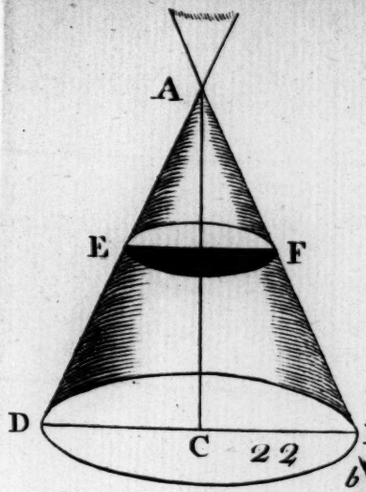


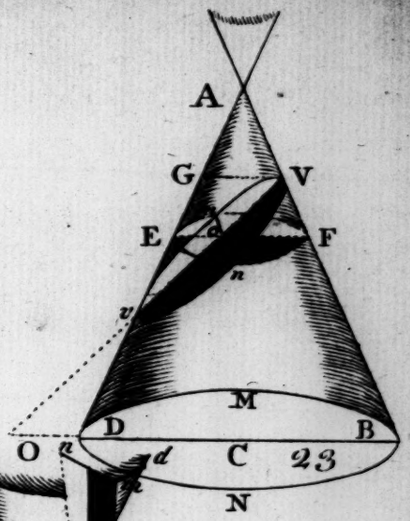


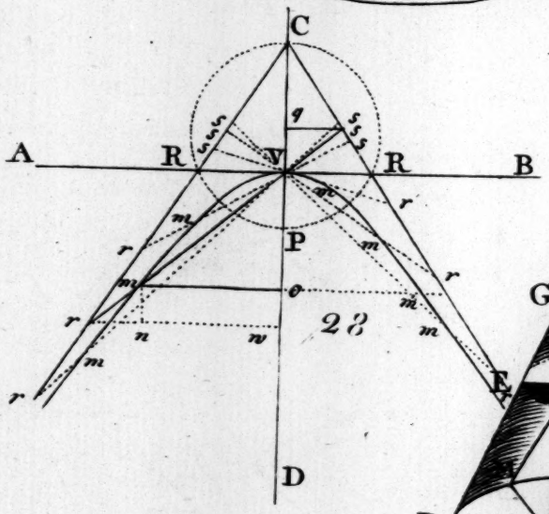

W. Wyeth Sculp.

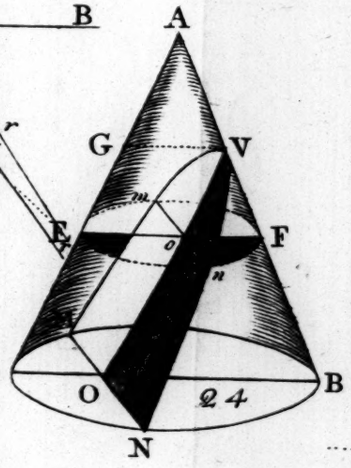
Plate. III.

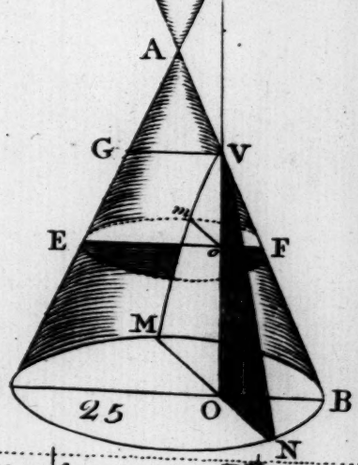


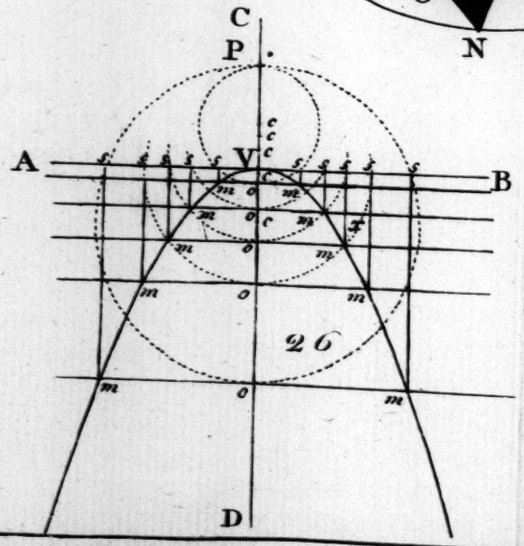



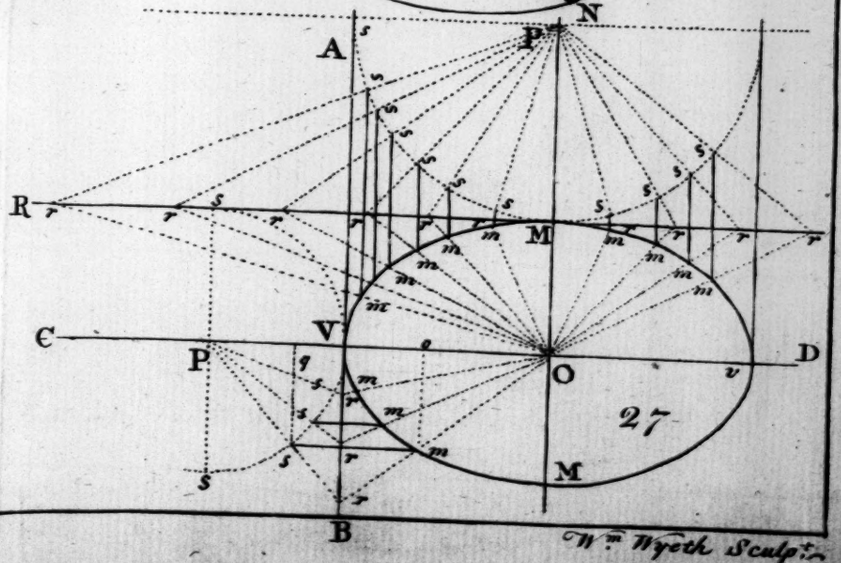




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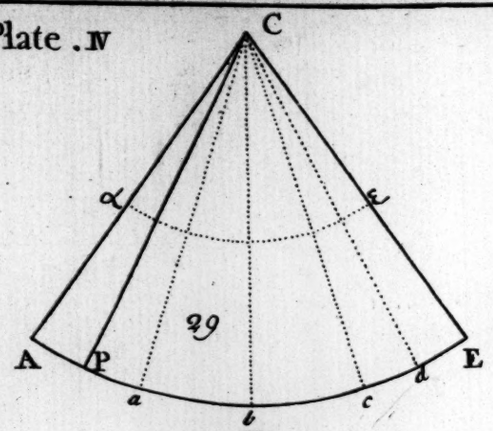
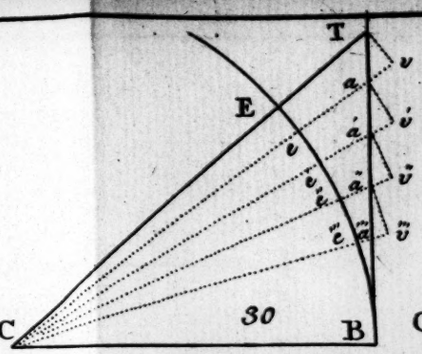
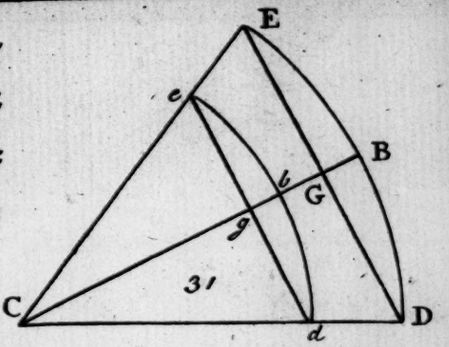
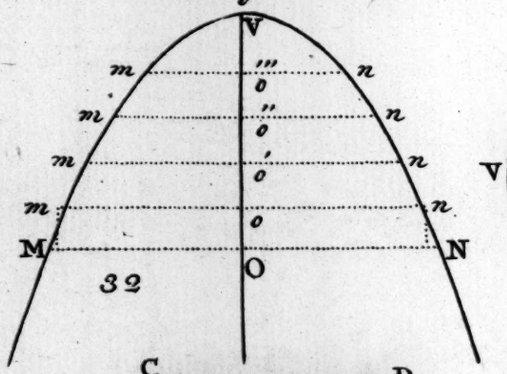
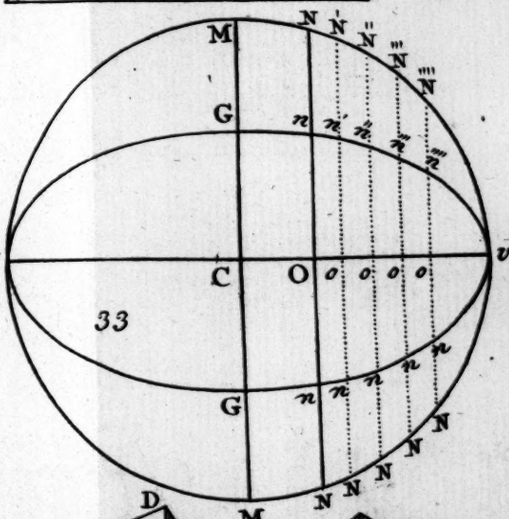
Plate . IV

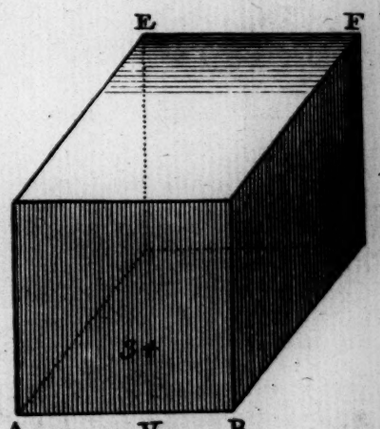


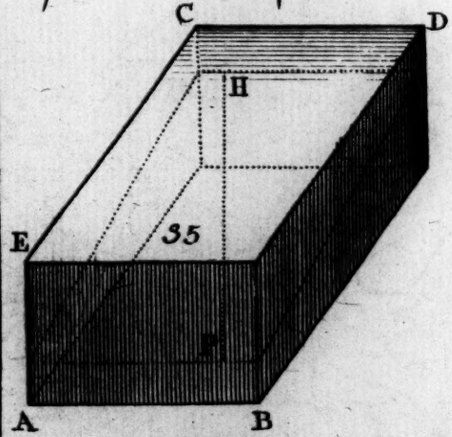



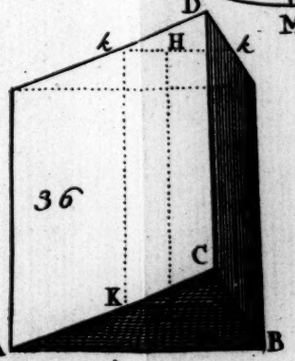



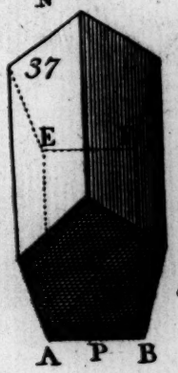



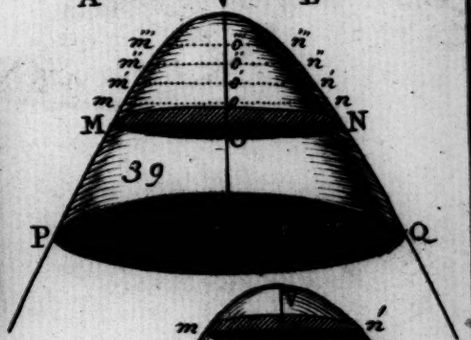



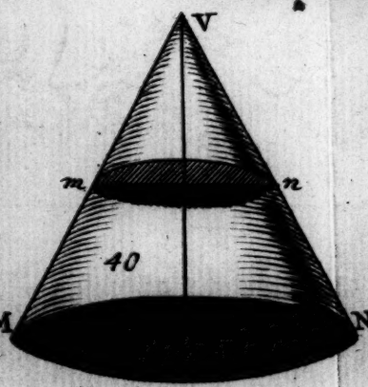



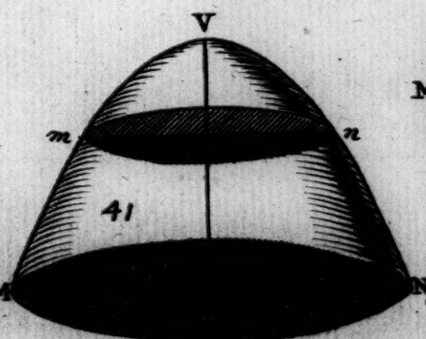



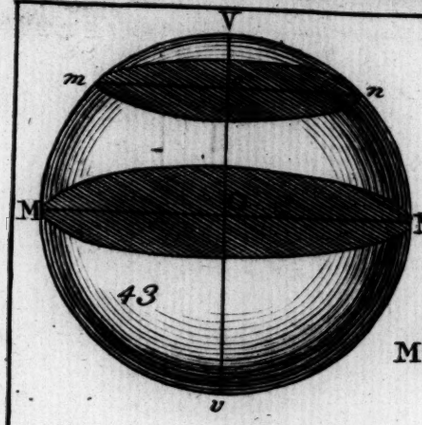



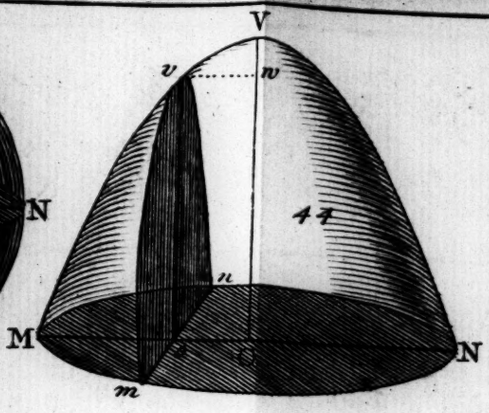


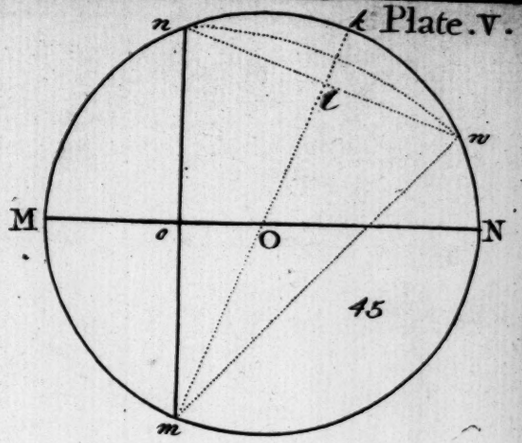



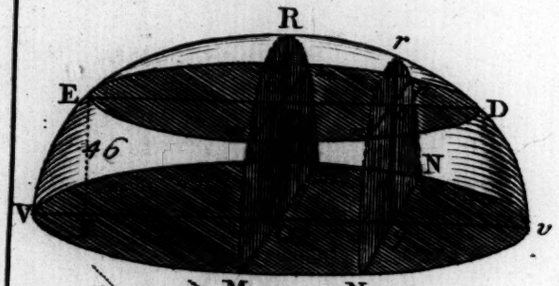
W: Wyeth Sculpt.

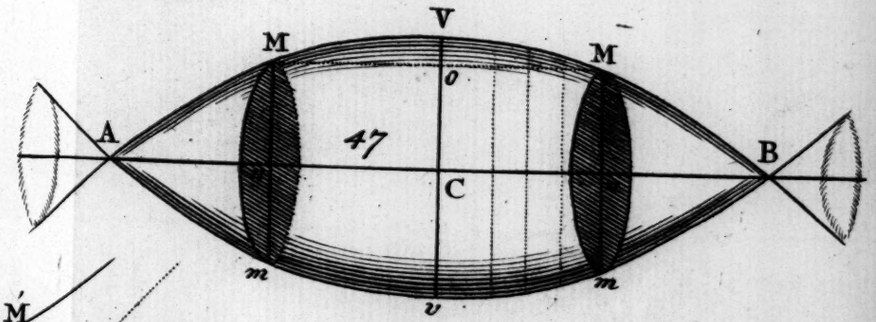
Plate. V.

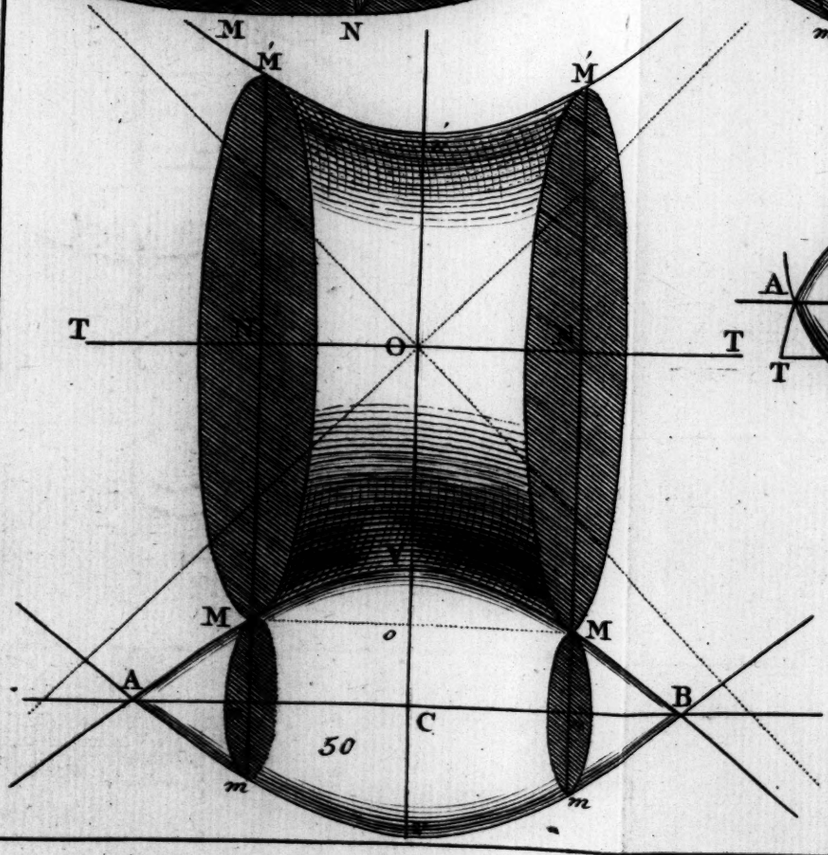


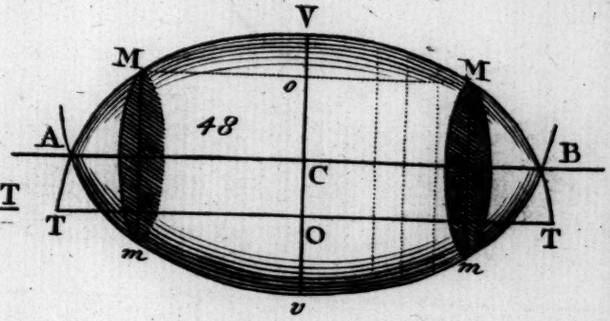



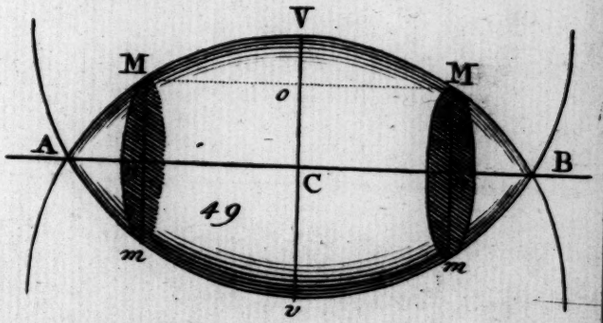



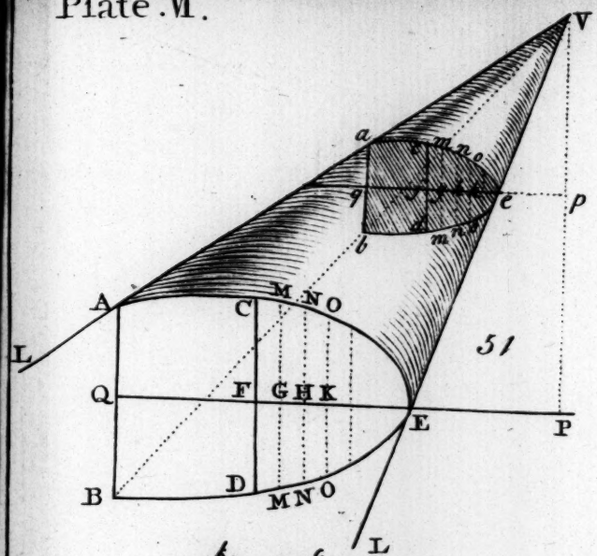




W. Wyeth Sculp.

Plate. VI.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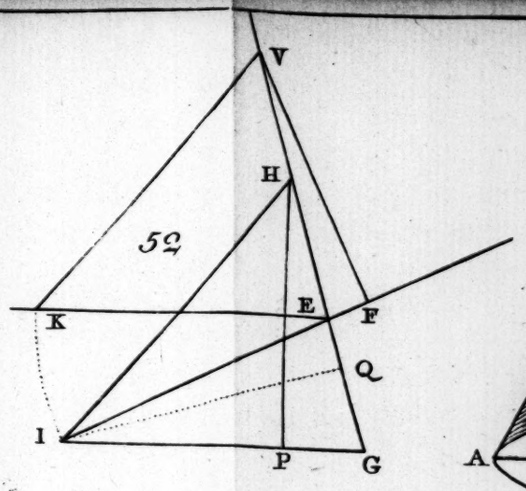
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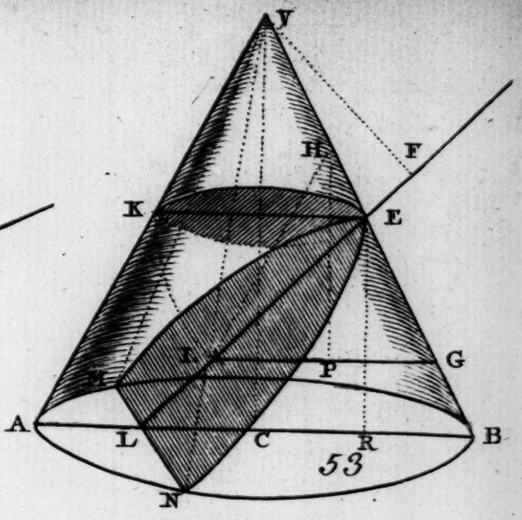
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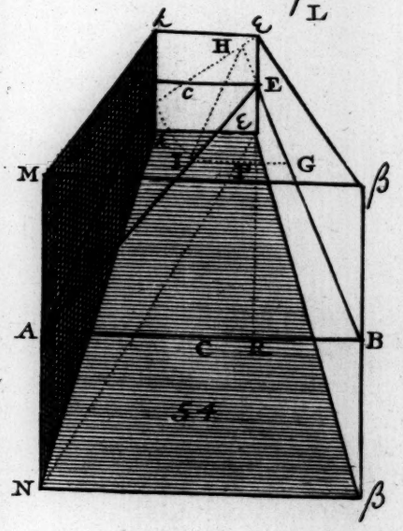
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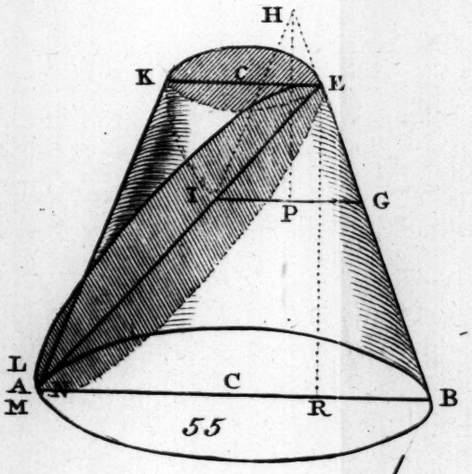
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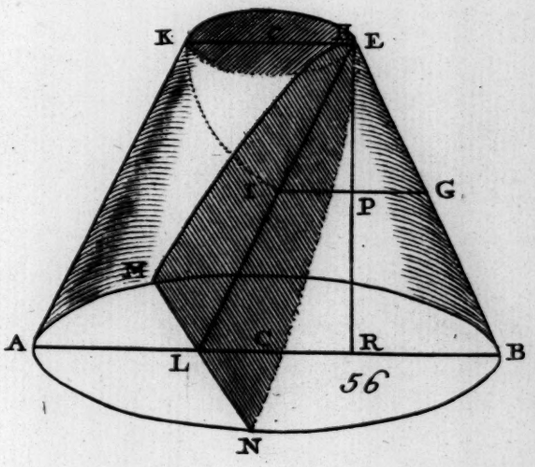
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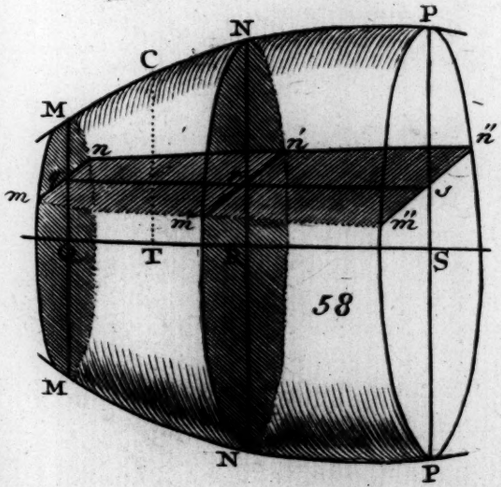
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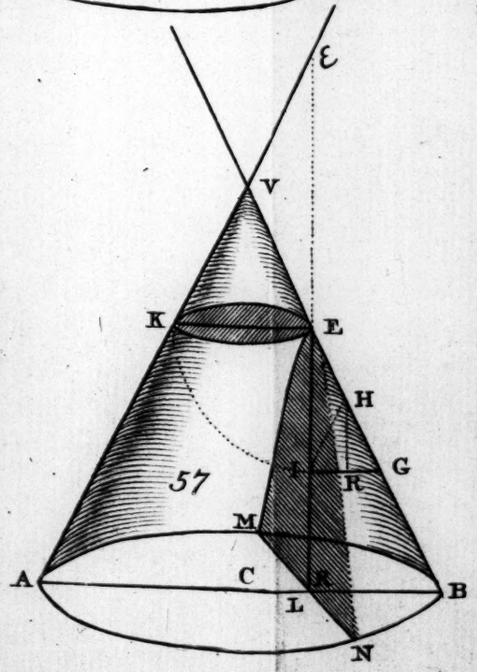
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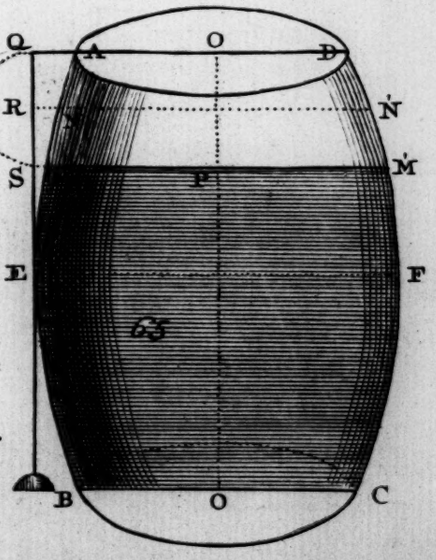
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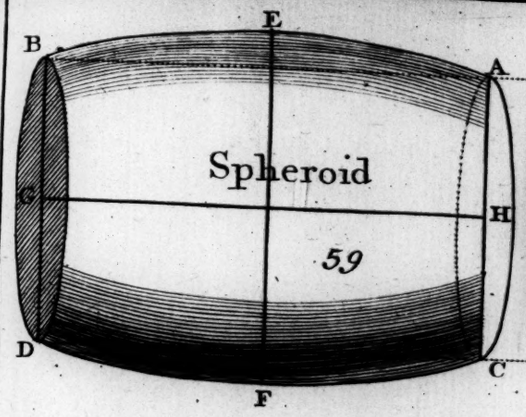


Spheroid

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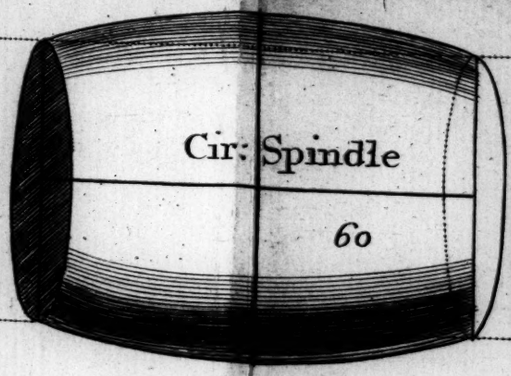

Cir: Spindle

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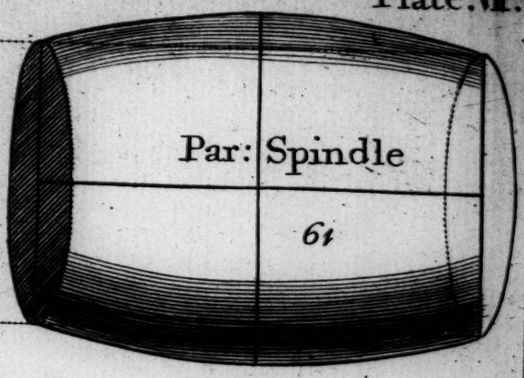
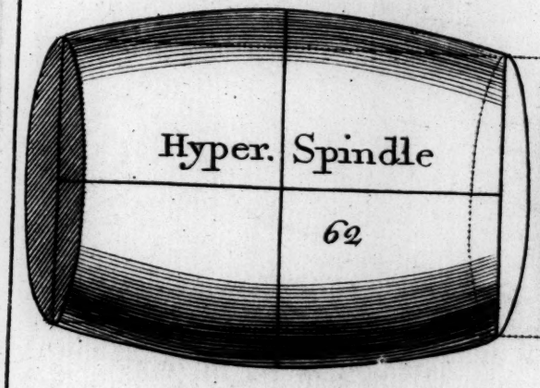


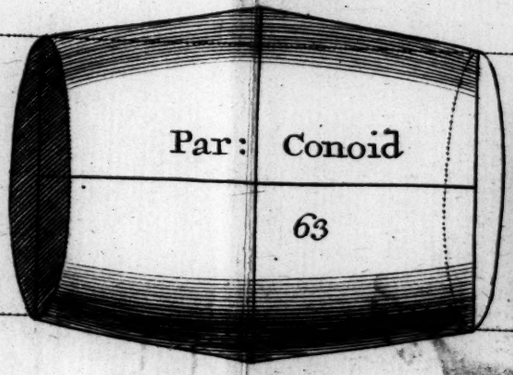
Par: Spindle

61



Hyper. Spindle

62



Par: Conoid

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Conic Fruſtum

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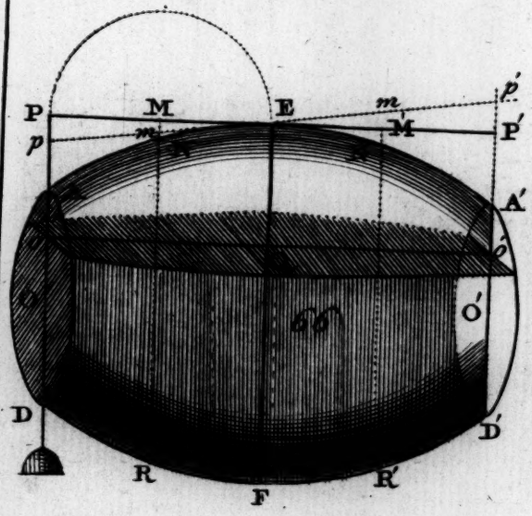


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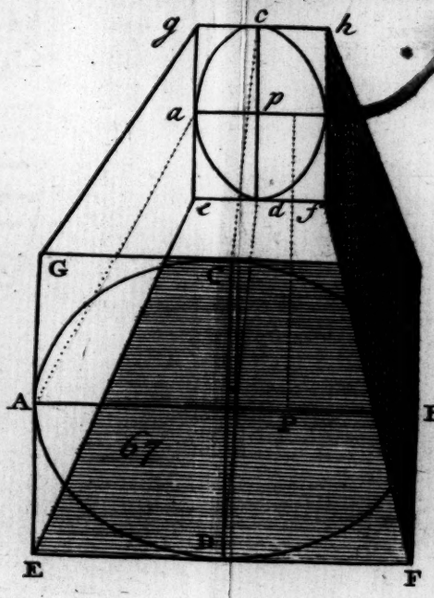


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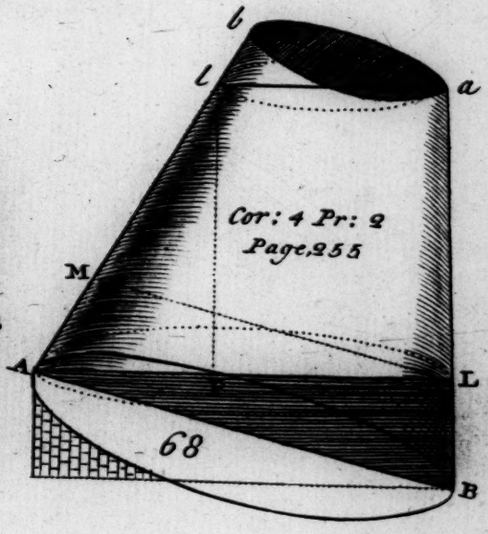


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W: Wyeth. Sculp.